

# Schoenberg matrices of radial positive definite functions and Riesz sequences in $L^2(\mathbb{R}^n)$ .

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## Abstract

Given a function  $f$  on the positive half-line  $\mathbb{R}_+$  and a sequence (finite or infinite) of points  $X = \{x_k\}_{k=1}^\omega$  in  $\mathbb{R}^n$ , we define and study matrices  $\mathcal{S}_X(f) = \|f(|x_i - x_j|)\|_{i,j=1}^\omega$  called Schoenberg's matrices. We are primarily interested in those matrices which generate bounded and invertible linear operators  $S_X(f)$  on  $\ell^2(\mathbb{N})$ . We provide conditions on  $X$  and  $f$  for the latter to hold. If  $f$  is an  $\ell^2$ -positive definite function, such conditions are given in terms of the Schoenberg measure  $\sigma(f)$ . We also approach Schoenberg's matrices from the viewpoint of harmonic analysis on  $\mathbb{R}^n$ , wherein the notion of the strong  $X$ -positive definiteness plays a key role. In particular, we prove that *each radial  $\ell^2$ -positive definite function is strongly  $X$ -positive definite* whenever  $X$  is separated. We also implement a “grammization” procedure for certain positive definite Schoenberg's matrices. This leads to Riesz–Fischer and Riesz sequences (Riesz bases in their linear span) of the form  $\mathcal{F}_X(f) = \{f(x - x_j)\}_{x_j \in X}$  for certain radial functions  $f \in L^2(\mathbb{R}^n)$ . Examples of Schoenberg's operators with various spectral properties are presented.

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# 1 Introduction

Positive definite functions have a long history, entering as an important chapter in all treatments of harmonic analysis. They can be traced back to papers of Carathéodory, Herglotz, Bernstein, culminating in Bochner’s celebrated theorem from 1932–1933. See definitions in Section 2.1.1.

In this paper we will be dealing primarily with radial positive definite functions (RPDF). RPDF’s have significant applications in probability theory, statistics, and approximation theory, where they occur as the characteristic functions or Fourier transforms of spherically symmetric probability distributions, the covariance functions of stationary and isotropic random fields, and the radial basis functions in scattered data interpolation. We denote the class of RPDF’s by  $\Phi_n$ .

We stick to the standard notation for the inner product  $(u, v)_n = (u, v) = u_1v_1 + \dots + u_nv_n$  of two vectors  $u = (u_1, \dots, u_n)$  and  $v = (v_1, \dots, v_n)$  in  $\mathbb{R}^n$ , and  $|u|_n = |u| = \sqrt{(u, u)}$  for the Euclidean norm of  $u$ . We want to emphasize from the outset that throughout the whole paper  $n$  is an arbitrary and fixed positive integer.

**Definition 1.1.** Let  $n \in \mathbb{N}$ . A real-valued and continuous function  $f$  on  $\mathbb{R}_+ = [0, \infty)$  is called a *radial positive definite function*, if for an arbitrary finite set  $\{x_1, \dots, x_m\}$ ,  $x_k \in \mathbb{R}^n$ , and  $\{\xi_1, \dots, \xi_m\} \in \mathbb{C}^m$

$$\sum_{k,j=1}^m f(|x_k - x_j|) \xi_j \bar{\xi}_k \geq 0. \quad (1.1)$$

The characterization of radial positive definite functions is a classical result due to I. Schoenberg [24, 25] (see, e.g., [2, Theorem 5.4.2]).

**Theorem 1.2.** A function  $f \in \Phi_n$ ,  $f(0) = 1$ , if and only if there exists a probability measure  $\nu$  on  $\mathbb{R}_+$  such that

$$f(r) = \int_0^\infty \Omega_n(rt) \nu(dt), \quad r \in \mathbb{R}_+, \quad (1.2)$$

where

$$\Omega_n(s) := \Gamma(q+1) \left(\frac{2}{s}\right)^q J_q(s) = \sum_{j=0}^\infty \frac{\Gamma(q+1)}{j! \Gamma(j+q+1)} \left(-\frac{s^2}{4}\right)^j, \quad q := \frac{n}{2} - 1, \quad (1.3)$$

$J_q$  is the Bessel function of the first kind and order  $q$ . Moreover,

$$\Omega_n(|x|) = \int_{S^{n-1}} e^{i(u,x)} \sigma_n(du), \quad x \in \mathbb{R}^n, \quad (1.4)$$

where  $\sigma_n$  is the normalized surface measure on the unit sphere  $S^{n-1} \subset \mathbb{R}^n$ .

The first three functions  $\Omega_n$ ,  $n = 1, 2, 3$ , can be computed as

$$\Omega_1(s) = \cos s, \quad \Omega_2(s) = J_0(s), \quad \Omega_3(s) = \frac{\sin s}{s}. \quad (1.5)$$

The main object under consideration in this paper arises from the definition of RPDF’s.

**Definition 1.3.** Let  $X = \{x_k\}_{k=1}^\omega \subset \mathbb{R}^n$  be a (finite or infinite) set of distinct points in  $\mathbb{R}^n$  and let  $f$  be a real-valued function defined on the right half-line  $\mathbb{R}_+$ . A matrix (finite or infinite)

$$\mathcal{S}_X(f) := \|f(|x_i - x_j|)\|_{i,j=1}^\omega, \quad \omega \leq \infty, \quad (1.6)$$

will be called a *Schoenberg matrix* generated by the set  $X$  and the function  $f$ . This function is referred to as the *Schoenberg symbol*.

It is clear that  $\mathcal{S}_X(f)$  is a Hermitian (real symmetric) matrix. By the definition, a function  $f \in \Phi_n$  if for each finite set  $X \subset \mathbb{R}^n$  the Schoenberg matrix  $\mathcal{S}_X(f)$  is nonnegative,  $\mathcal{S}_X(f) \geq 0$ .

We undertake a detailed study of Schoenberg's matrices from two different points of view. The first one, considered in Section 3, comes from operator theory.

If the columns of  $\mathcal{S}_X(f)$  are in  $\ell^2 := \ell^2(\mathbb{N})$ , then one can associate a minimal symmetric operator  $S_X(f)$  with  $\mathcal{S}_X(f)$  in a natural way. We call it a *Schoenberg operator*. If  $S_X(f)$  appears to be bounded, a matrix  $\mathcal{S}_X(f)$  (admitting some abuse of language) will be called bounded. *The first main goal of the paper is to find necessary and sufficient conditions on  $X$  and  $f$ , which ensure that the matrix  $\mathcal{S}_X(f)$  is bounded.* We also suggest conditions on  $X$  and  $f$  for  $S_X(f)$  to be invertible, i.e., to have a bounded inverse.

Throughout the paper we always assume that  $X$  is a *separated set*, i.e.,

$$d_* = d_*(X) := \inf_{i \neq j} |x_i - x_j| > 0, \quad (1.7)$$

(the term *uniformly discrete* is also in common usage). We denote by  $\mathcal{X} = \mathcal{X}_n$  the class of all separated sets  $X \subset \mathbb{R}^n$  and by  $\mathcal{L} = \mathcal{L}(X)$  a linear span of  $X$ , a subspace in  $\mathbb{R}^n$  of dimension  $d = d(X) = \dim \mathcal{L} \leq n$ . With no loss of generality we can assume that  $x_1 = 0$ .

Next, denote by  $\mathcal{M}_+$  the following class of functions:

$$f \in \mathcal{M}_+ : \quad f \geq 0, \quad f \downarrow, \quad f(0) = 1. \quad (1.8)$$

With this preparation our main result on boundedness of  $\mathcal{S}_X(f)$  reads as follows.

**Theorem 1.4.** *Let  $f \in \mathcal{M}_+$ ,  $X \in \mathcal{X}_n$  and let  $d = \dim \mathcal{L}(X)$ .*

(i) *If  $t^{d-1}f(\cdot) \in L^1(\mathbb{R}_+)$ , then the Schoenberg matrix  $\mathcal{S}_X(f)$  is bounded on  $\ell^2$  and*

$$\|\mathcal{S}_X(f)\| \leq 1 + d^2 \left( \frac{5}{d_*(X)} \right)^d \int_0^\infty t^{d-1} f(t) dt. \quad (1.9)$$

(ii) *Moreover,  $S_X(f)$  has a bounded inverse whenever, in addition,*

$$d_*(X) > 5d^{2/d} \|t^{d-1}f\|_{L^1(\mathbb{R}_+)}^{1/d}. \quad (1.10)$$

(iii) *Conversely, let  $S_Y(f)$  be bounded for at least one  $\delta$ -regular set  $Y$ . Then  $t^{d-1}f(\cdot) \in L^1(\mathbb{R}_+)$ .*

Concerning regular sets see Definition 3.3. For instance,  $X = \delta\mathbb{Z}^n$  is  $\delta$ -regular.

In particular, Theorem 1.4 completely describes bounded operators  $S_X(f)$  with symbols  $f$  from the classes  $\Phi_\infty(\alpha)$  defined below in Section 2.1.3.

We also discuss the Fredholm property of the Schoenberg operators, precisely, the case when  $S_X(f) = I + T$ ,  $T$  is a compact operator on  $\ell^2$ .

An interesting example of general Schoenberg operators arises when the set  $X$  is a Toeplitz sequence, that is,  $|x_i - x_j| = |i - j|$  for all  $i, j \in \mathbb{N}$ . Such operators will be called the *Schoenberg–Toeplitz operators*. We obtain necessary and sufficient conditions for the Schoenberg–Toeplitz operators with special symbols to be bounded and describe their spectra in terms of Schoenberg’s symbols. We show that such (possibly unbounded) operators are always self-adjoint.

Our second viewpoint on Schoenberg’s matrices is related to harmonic analysis on  $\mathbb{R}^n$ .

The main result of Section 4 is related to the notion of the strong  $X$ -positive definiteness.

**Definition 1.5.** Let  $f \in \Phi_n$  and  $X = \{x_k\}_{k \in \mathbb{N}} \subset \mathbb{R}^n$ . We say that  $f$  is *strongly  $X$ -positive definite* (or the Schoenberg matrix  $\mathcal{S}_X(f)$  is positive definite) if for each set  $\xi = \{\xi_1, \dots, \xi_m\} \in \mathbb{C}^m \setminus \{0\}$  and any finite set  $\{x_j\}_{j=1}^m$  of distinct points  $x_j \in X$  there exists a constant  $c = c(X) > 0$ , independent of  $\xi$  and  $m$  such that

$$\sum_{k,j=1}^m f(|x_k - x_j|) \xi_j \overline{\xi_k} \geq c \sum_{k=1}^m |\xi_k|^2. \quad (1.11)$$

The same definition with obvious changes applies to general (not necessarily radial) positive definite functions.

We say that  $f$  is *strictly  $X$ -positive definite* if for each  $m \in \mathbb{N}$  and  $\xi = \{\xi_1, \dots, \xi_m\} \in \mathbb{C}^m \setminus \{0\}$  inequality (1.11) holds with  $c = 0$ .

Equivalently,  $f$  is strictly  $X$ -positive definite, if for any finite subset  $Y \subset X$  the Schoenberg matrix  $\mathcal{S}_Y(f)$  is non-singular, i.e., the minimal eigenvalue  $\lambda_{\min}(\mathcal{S}_Y(f))$  of  $\mathcal{S}_Y(f)$  is positive, and strongly  $X$ -positive definite, if  $\mathcal{S}_Y(f)$  are “uniformly positive definite”, that is,

$$\inf_{Y \subset X} \lambda_{\min}(\mathcal{S}_Y(f)) > 0,$$

where the infimum is taken over all finite subsets  $Y \subset X$ .

The notion of strong  $X$ -positive definiteness makes sense for any  $f \in \Phi_n$  regardless of whether the Schoenberg operator  $S_X(f)$  is defined or not. In the former case *the strong  $X$ -positive definiteness of  $f$  is identical to positive definiteness of  $S_X(f)$* , i.e., validity of the inequality

$$(S_X(f)h, h) \geq \varepsilon |h|^2, \quad h \in \text{dom } S_X(f) \subset \ell^2, \quad \varepsilon > 0. \quad (1.12)$$

with some  $\varepsilon > 0$  independent of  $h$ . So Definition 1.5 merely extends a property (1.12) of  $S_X(f)$ , when the latter exists, to the case of an arbitrary Schoenberg matrix  $\mathcal{S}_X(f)$ , not necessarily generating an operator in  $\ell^2$ .

Each strongly  $X$ -positive definite function  $f$  is also strictly  $X$ -positive definite. For finite sets  $X$  both notions are equivalent due to the compactness of the balls in  $\mathbb{C}^m$ . The following problem seems to be important and difficult.

**Problem I.** Let  $f$  be a radial positive definite function on  $\mathbb{R}^n$ . Characterize those countable subsets  $X$  of  $\mathbb{R}^n$  for which  $f$  is strongly  $X$ -positive definite.

It was proved in [27] (see also [14, Theorem 3.6]) that each function  $f \in \Phi_n$ ,  $n \geq 2$ , is strictly  $X$ -positive definite for any set  $X$  of distinct points in  $\mathbb{R}^n$ . This fact has been heavily exploited in [14] for investigation of certain spectral properties of  $2D$  and  $3D$  Schrödinger operator with a *finite number* of point interactions. On the other hand, if a radial positive definite function is  $X$ -strongly positive definite, then  $X$  is necessarily separated (see Proposition 3.21).

Our *second main goal is to give a partial solution to Problem I*. Heading to the solution of this problem we prove the following result.

**Theorem 1.6.** *Let  $(\text{const} \neq)f \in \Phi_n$ ,  $n \geq 2$ , with the representing measure  $\nu = \nu(f)$  from (1.2). If  $\nu$  is equivalent to the Lebesgue measure on  $\mathbb{R}_+$ , then  $f$  is strongly  $X$ -positive definite for each  $X \in \mathcal{X}_n$ .*

Actually, the most complete result on the strong  $X$ -positivity and the boundedness of  $S_X(f)$  is obtained for the class  $\Phi_\infty := \bigcap_{n \in \mathbb{N}} \Phi_n$  and its subclasses  $\Phi_\infty(\alpha)$ ,  $\alpha \in (0, 2]$  defined in the next section. It looks as follows.

**Theorem 1.7.** *Let  $f \in \Phi_\infty(\alpha)$ ,  $0 < \alpha \leq 2$  and  $X \in \mathcal{X}_n$ . Then*

- (i)  *$f$  is strongly  $X$ -positive definite. In particular, if  $\mathcal{S}_X(f)$  generates an operator  $S_X(f)$  on  $\ell^2$ , then it is positive definite and so invertible.*
- (ii) *If the Schoenberg measure  $\sigma = \sigma_f$  in (2.6) satisfies*

$$\int_0^\infty s^{-\frac{d}{\alpha}} \sigma(ds) < \infty, \quad d = \dim \mathcal{L}(X), \quad (1.13)$$

*then the Schoenberg matrix  $\mathcal{S}_X(f)$  generates a bounded (necessarily invertible) operator.*

- (iii) *Conversely, let  $S_Y(f)$  be bounded for at least one  $\delta$ -regular set  $Y$ . Then (1.13) holds.*

The concept of “grammization” plays a key role in the rest of the Section 4.

It is a common knowledge that every positive matrix is the Gramm matrix of a certain system of vectors

$$\mathcal{A} = \|a_{ij}\|_{i,j \in \mathbb{N}} \geq 0 \Leftrightarrow \mathcal{A} = \|(\varphi_i, \varphi_j)\|_{i,j \in \mathbb{N}} =: Gr(\{\varphi_k\}_{k \in \mathbb{N}}, \mathcal{H}) \quad (1.14)$$

$\{\varphi_k\}_{k \in \mathbb{N}}$  are vectors in a Hilbert space  $\mathcal{H}$ . According to the classical result of Bari, the property of a Gramm matrix  $Gr\{\varphi_k\}_{k \in \mathbb{N}}$  to generate a bounded and invertible operator on  $\ell^2$  amounts to the sequence  $\{\varphi_k\}_{k \in \mathbb{N}}$  to be a Riesz sequence (Riesz basis in its linear span).

The main applications of Theorems 1.4 and 1.6 are based on the grammization procedure and concern Riesz–Fischer and Riesz sequences of shifts  $\mathcal{F}_X(f) = \{f(\cdot - x_j)\}_{j \in \mathbb{N}}$ ,  $X = \{x_j\}_{j \in \mathbb{N}} \subset \mathbb{R}^n$ , of certain radial functions  $f \in L^2(\mathbb{R}^n)$ .

**Proposition 1.8.** *Let  $f \in L^2(\mathbb{R}^n)$  be a real-valued and radial function such that its Fourier transform  $\widehat{f} \neq 0$  a.e., and  $X = \{x_j\}_{j \in \mathbb{N}} \subset \mathbb{R}^n$ . Then the following statements are equivalent.*

- (i)  *$\mathcal{F}_X(f)$  forms a Riesz–Fischer sequence in  $L^2(\mathbb{R}^n)$ ;*
- (ii)  *$\mathcal{F}_X(f)$  is uniformly minimal in  $L^2(\mathbb{R}^n)$ ;*
- (iii)  *$X$  is a separated set, i.e.,  $d_*(X) > 0$ .*

**Theorem 1.9.** *Let  $f \in L^2(\mathbb{R}^n)$  be a real-valued and radial function such that its Fourier transform  $\widehat{f} \neq 0$  a.e. and  $X = \{x_j\}_{j \in \mathbb{N}} \subset \mathbb{R}^n$ . Let  $F$  and  $F_0$  be defined as*

$$F(t) = (2\pi)^{n/2} |\widehat{f}(t)|^2 = F_0(|t|), \quad \widehat{F}(t) = \widetilde{F}_0(|t|), \quad (1.15)$$

*and assume that for some majorant  $h \in \mathcal{M}_+$  (1.8) the relations*

$$|\widetilde{F}_0(s)| \leq h(s), \quad s^{n-1}h(s) \in L^1(\mathbb{R}_+) \quad (1.16)$$

*hold. Then the following statements are equivalent.*

- (i)  $\mathcal{F}_X(f)$  forms a Riesz sequence in  $L^2(\mathbb{R}^n)$ ;
- (ii)  $\mathcal{F}_X(f)$  forms a basis in its linear span;
- (iii)  $\mathcal{F}_X(f)$  is uniformly minimal in  $L^2(\mathbb{R}^n)$ ;
- (iv)  $X$  is a separated set, i.e.,  $d_*(X) > 0$ .

The idea of the proof is related to the fact that the system  $\mathcal{F}_X(f)$  performs the grammization of a certain Schoenberg's matrix. So once we show that the latter generates a bounded and invertible operator on  $\ell^2$ , the result is immediate from the Bari theorem. Thereby we make up a bridge between Riesz sequences and Gramm matrices on the one hand and Schoenberg's matrices and operators on the other hand.

We consider a number of examples which satisfy the assumptions of Proposition 1.8 and Theorem 1.9. Among them

$$f(x) = f_a(x) = e^{-a|x|^2}, \quad f(x) = f_{a,\mu}(x) = \left(\frac{a}{|x|}\right)^\mu K_\mu(a|x|), \quad (1.17)$$

where  $K_\mu$  is the modified Bessel function of the second kind and order  $\mu$ ,  $0 \leq \mu < n/4$ .

Let us emphasize, that our choice of the second system in (1.17) is also motivated by applications to elliptic operators with point interactions, since the functions  $f_{a,\mu}(\cdot - x_j)$  occur naturally in the spectral theory of such operators for certain other values of  $\mu$ . We hope to continue the study of this subject in our forthcoming papers.

It is worth stressing that in the abstract setting the uniform minimality is much weaker than the Riesz sequence property. Nonetheless the equivalence of these properties is well-known for certain classical systems:

- (i) Exponential system  $\{e^{i\lambda_k x}\}_{\lambda_k \in \Lambda}$  in  $L^2[0, a)$ ,  $a \leq \infty$ , provided that  $\inf_k (\Im \lambda_k) > -\infty$ .
- (ii) The system of rational functions  $\{(1 - |\lambda_k|^2)^{1/2}(1 - \lambda_k z)^{-1}\}_{\lambda_k \in \Lambda}$  in  $L^2(\mathbb{T})$ .

In the forthcoming paper [13] we shed light on this effect and show that a transparent connection of the result in Theorem 1.9 with the corresponding property of the system of exponential functions is not occasional and has deeper reasons.

From the very starting point we were influenced by the paper [17], wherein a tight connection between the spectral theory of 3D Schrödinger operators with *infinitely many* point interactions and RPDF's in  $\mathbb{R}^3$  was discovered and exploited in both directions. In particular, a special case of Theorem 1.7 (for  $n = d = 3$  and  $\alpha = 1$ ) was proved in [17] by applying machinery of the spectral theory and the grammization of the Schoenberg–Bernstein matrix  $\mathcal{S}_X(e^{-as})$ , which is achieved for  $n = 3$  by the system

$$f_{a,1/2}(x - x_j) = \sqrt{\frac{a}{|x - x_j|}} K_{1/2}(a|x - x_j|) = \sqrt{\frac{\pi}{2}} \frac{e^{-a|x - x_j|}}{|x - x_j|}, \quad j \in \mathbb{N},$$

(see (4.29)). However the spectral methods applied in [17] *cannot be extended to either*  $n \geq 4$  *or*  $\alpha \neq 1$ . Our reasoning is based on the harmonic and Fourier analysis on  $\mathbb{R}^n$  and works for an arbitrary dimension  $n \geq 2$ .

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## 2 Preliminaries

### 2.1 Positive definite functions

Recall some basic facts and notions related to positive definite functions [2, 5, 28, 31].

**Definition 2.1.** A function  $g : \mathbb{R}^n \rightarrow \mathbb{C}$  is called *positive definite* if  $g$  is continuous at the origin and for arbitrary finite sets  $\{x_1, \dots, x_m\}$ ,  $x_k \in \mathbb{R}^n$  and  $\{\xi_1, \dots, \xi_m\} \in \mathbb{C}^m$  we have

$$\sum_{k,j=1}^m g(x_k - x_j) \xi_j \bar{\xi}_k \geq 0. \quad (2.1)$$

The set of positive definite function on  $\mathbb{R}^n$  is denoted by  $\Phi(\mathbb{R}^n)$ . Clearly, a function  $g \in \Phi(\mathbb{R}^n)$  if and only if it is continuous at the origin, and the matrix  $\mathcal{B}_X(g) := \|g(x_k - x_j)\|_{k,j=1}^m$  is nonnegative,  $\mathcal{B}_X(g) \geq 0$ , for all finite subsets  $X = \{x_j\}_{j=1}^m$  in  $\mathbb{R}^n$ .

A celebrated theorem of S. Bochner [8] gives a description of the class  $\Phi(\mathbb{R}^n)$ .

**Theorem 2.2.** *A function  $g$  is positive definite on  $\mathbb{R}^n$  if and only if there exists a finite positive Borel measure  $\mu$  on  $\mathbb{R}^n$  such that*

$$g(x) = \int_{\mathbb{R}^n} e^{i(u,x)} \mu(du), \quad x \in \mathbb{R}^n. \quad (2.2)$$

When  $g$  is a radial function,  $g(\cdot) = f(|\cdot|)$ ,  $f \in \Phi_n$ , the representing measure  $\nu$  in (1.2) is related to the Bochner measure  $\mu$  by  $\nu\{[0, r]\} = \mu\{|x| \leq r\}$  (cf. [2, Section V.4.2]).

#### 2.1.1 Class $\Phi_\infty$ of radial positive definite functions

Going over to the classes  $\Phi_n$  of PRDF's, note that the sequence  $\{\Phi_n\}_{n \in \mathbb{N}}$  is known to be nested, i.e.,  $\Phi_{n+1} \subset \Phi_n$ , and inclusion is proper (see [24], [28, Section 6.3]). So the intersection  $\Phi_\infty = \bigcap_{n \in \mathbb{N}} \Phi_n$  comes in naturally. The class  $\Phi_\infty$  is the case of study in the pioneering paper of I. Schoenberg [24]. According to the Schoenberg theorem (see, e.g., [2, Theorem 5.4.3]),  $f \in \Phi_\infty$ ,  $f(0) = 1$ , if and only if it admits an integral representation

$$f(t) = \int_0^\infty e^{-st^2} \sigma(ds), \quad t \geq 0, \quad (2.3)$$

with  $\sigma$  being a probability measure on  $\mathbb{R}_+$ . The measure  $\sigma$ , which is called a *Schoenberg measure* of  $f \in \Phi_\infty$ , is then uniquely determined by  $f$ .

Another characterization of the class  $\Phi_\infty$  is  $\Phi_\infty = \Phi(\ell^2)$ , where the latter is the class of radial positive definite functions on the real Hilbert space  $\ell^2$  (see, e.g., [28, p.283]). Indeed, since  $\mathbb{R}^n$  is embedded in  $\ell^2$  for each  $n \in \mathbb{N}$ , we have  $\Phi(\ell^2) \subset \Phi_\infty$ . Conversely, let  $f \in \Phi_\infty$  and  $Y = \{y_k\}_{k=1}^m \subset \ell^2$ ,  $y_k = (y_{k1}, y_{k2}, \dots)$ . Define truncations  $y_k^{(n)} := (y_{k1}, y_{k2}, \dots, y_{kn}, 0, 0, \dots) \in \mathbb{R}^n$ . Then for each  $n$

$$\|f(|y_i^{(n)} - y_j^{(n)}|)\|_{i,j=1}^m \geq 0.$$

As  $\lim_{n \rightarrow \infty} |y_i^{(n)} - y_j^{(n)}| = |y_i - y_j|$  and  $f$  is continuous, the matrix  $\|f(|y_i - y_j|)\|_{i,j=1}^m$  is also positive definite, as claimed.

### 2.1.2 Bernstein class $CM(\mathbb{R}_+)$ of absolute monotone functions

**Definition 2.3.** A function  $f \in C(\mathbb{R}_+)$  is called *completely monotone* if

$$(-1)^k f^{(k)}(t) \geq 0, \quad t > 0, \quad k = 0, 1, 2, \dots \quad (2.4)$$

The set of such functions is denoted by  $CM(\mathbb{R}_+)$ . A function  $f$  belongs to a subclass  $CM_0(\mathbb{R}_+)$  of  $CM(\mathbb{R}_+)$  if  $f \in CM(\mathbb{R}_+)$  and  $f(+0) = 1$ .

A fundamental theorem of S. Bernstein – D. Widder ([6, 33], see also [2, p.204]) claims that  $f \in CM(\mathbb{R}_+)$  if and only if there exists a positive Borel measure  $\tau$  on  $\mathbb{R}_+$  such that

$$f(t) = \int_0^\infty e^{-st} \tau(ds), \quad t > 0. \quad (2.5)$$

The measure  $\tau$ , which is called a *Bernstein measure* of  $f \in CM(\mathbb{R}_+)$ , is then uniquely determined by  $f$ .  $\tau$  is the probability measure if and only if  $f \in CM_0(\mathbb{R}_+)$ .

### 2.1.3 Subclasses $\Phi_\infty(\alpha)$ of radial positive definite functions

By definition, a class  $\Phi_\infty(\alpha)$  consists of functions which admit an integral representation

$$f(t) = \int_0^\infty e^{-st^\alpha} \sigma(ds), \quad t \geq 0, \quad 0 < \alpha \leq 2, \quad (2.6)$$

$\sigma$  is a probability measure on  $\mathbb{R}_+$ . We call the functions  $f \in \Phi_\infty(\alpha)$   $\alpha$ -stable. They are tightly related to  $\alpha$ -stable distributions in probability theory. So,  $\Phi_\infty(2) = \Phi_\infty$ ,  $\Phi_\infty(1) = CM_0(\mathbb{R}_+)$ . The classes  $\Phi_\infty(\alpha)$  are known to admit the following characterization [7]:  $f \in \Phi_\infty(\alpha)$ ,  $0 < \alpha \leq 2$ , if and only if the function  $f(|x|_\alpha)$  is positive definite, where

$$x = (x_1, x_2, \dots), \quad |x|_\alpha := \left( \sum_{n=1}^\infty |x_n|^\alpha \right)^{\frac{1}{\alpha}}.$$

Note that the family  $\{\Phi_\infty(\alpha)\}_{0 < \alpha \leq 2}$  is nested, i.e.,

$$\Phi_\infty(\alpha_1) \subset \Phi_\infty(\alpha_2), \quad 0 < \alpha_1 < \alpha_2 \leq 2, \quad (2.7)$$

and the inclusion is proper (see, e.g., [7, 11]). Indeed, (2.7) is equivalent to

$$\Phi_\infty(\alpha) \subset \Phi_\infty(1) = CM_0(\mathbb{R}_+), \quad 0 < \alpha < 1, \quad (2.8)$$

(a simple change of variables under the integral sign). Next, it is known (and can be easily verified by induction, using Leibniz chain rule) that the function  $f = e^{-g} \in CM(\mathbb{R}_+)$  provided  $g' \in CM(\mathbb{R}_+)$ . Hence

$$\exp(-sx^\alpha) \in CM_0(\mathbb{R}_+), \quad 0 < \alpha \leq 1,$$

so (2.4) holds for this function. Differentiation under the integral sign shows that the same is true for each  $f \in \Phi_\infty(\alpha)$  and (2.8) follows. The same argument implies  $\exp(-sx^\beta) \notin \Phi_\infty(\alpha)$  for  $\beta > \alpha$ .

For the detailed account of the subject see, e.g., [30, Chapter 2.7].



## 2.2 Infinite matrices and Schur test

We say that an infinite matrix  $\mathcal{A} = \|a_{kj}\|_{k,j \in \mathbb{N}}$  with complex entries  $a_{kj}$  generates a bounded linear operator  $A$  on the Hilbert space  $\ell^2 = \ell^2(\mathbb{N})$  (or simply that an infinite matrix is a bounded operator on  $\ell^2$ ) if there exists a bounded linear operator  $A$  such that

$$\langle Ax, y \rangle = \sum_{k,j=1}^{\infty} a_{kj} x_k \overline{y_j}, \quad x = \{x_k\}_{k \in \mathbb{N}}, \quad y = \{y_k\}_{k \in \mathbb{N}}, \quad x, y \in \ell^2. \quad (2.9)$$

Clearly, if  $\mathcal{A}$  defines a bounded operator  $A$ , then  $A$  is uniquely determined by equalities (2.9).

The following result known as the *Schur test* (due in substance to I. Schur) provides certain general conditions for an infinite matrix  $\mathcal{A} = \|a_{ij}\|_{i,j \in \mathbb{N}}$  to define a bounded linear operator  $A$  on  $\ell^2$  (see, e.g., [19, Theorem 5.2.1]). One of the simplest its versions can be stated as follows.

**Lemma 2.4.** *Let  $\mathcal{A} = \|a_{ij}\|_{i,j \in \mathbb{N}}$  be an infinite Hermitian matrix which satisfies*

$$C := \sup_{j \in \mathbb{N}} \sum_{i=1}^{\infty} |a_{ij}| < \infty. \quad (2.10)$$

*Then  $\mathcal{A}$  defines a bounded self-adjoint operator  $A$  on  $\ell^2$  with  $\|A\| \leq C$ .*

Note that the Schur test applies to general (not necessarily Hermitian) matrices with two independent conditions for their rows and columns

$$C_1 := \sup_{j \in \mathbb{N}} \sum_{i=1}^{\infty} |a_{ij}| < \infty, \quad C_2 := \sup_{i \in \mathbb{N}} \sum_{j=1}^{\infty} |a_{ij}| < \infty,$$

and the bound for the norm is  $\|A\|^2 \leq C_1 C_2$ .

The condition for compactness of  $A$  is similar.

**Lemma 2.5.** *Suppose that*

$$\delta_p := \sup_{j \geq p} \sum_{k \geq p} |a_{jk}| < \infty, \quad \forall p \in \mathbb{N}, \quad \text{and} \quad \lim_{p \rightarrow \infty} \delta_p = 0. \quad (2.11)$$

*Then the Hermitian matrix  $\mathcal{A} = \|a_{kj}\|_{k,j \in \mathbb{N}}$  generates a compact self-adjoint operator on  $\ell^2$ .*

For the proof see, e.g., [17, Lemma 2.23]

## 3 Schoenberg matrices from operator theory viewpoint

### 3.1 Bounded Schoenberg operators

Sometimes an infinite Schoenberg matrix generates a bounded linear operator  $S_X(f)$  on  $\ell^2$ . We call  $S_X(f)$  a *Schoenberg operator*. The main problem we address here concerns conditions on the test set  $X \subset \mathbb{R}^n$  and the Schoenberg symbol  $f$  for  $S_X(f)$  to be bounded.

We will be dealing primarily with separated sets  $X$ ,

$$d_* = d_*(X) := \inf_{i \neq j} |x_i - x_j| > 0.$$

Recall the notation  $\mathcal{X}_n$  for the class of all separated sets in  $\mathbb{R}^n$  and  $\mathcal{L} = \mathcal{L}(X)$  for the linear span of  $X$ ,  $d = \dim \mathcal{L} \leq n$ .

The result below gives an upper bound for the number of points of a separated set  $X$  in a spherical layer

$$U_r(p, q, a, X) := \{y \in \mathcal{L}(X) : pr \leq |y - a| < qr\}, \quad q > p \geq 0,$$

centered at  $a \in \mathcal{L}(X)$ .

**Lemma 3.1.** *Let  $X = \{x_k\}_{k \in \mathbb{N}} \in \mathcal{X}_n$ ,  $d_*(X) = \varepsilon > 0$ , and let  $a \in \mathcal{L}(X)$ . Then for the number  $N_m(X)$  of the points  $\{x_j\}$  contained in  $U_\varepsilon(m, m+1, a, X)$ ,  $m = 0, 1, \dots$ , the inequality*

$$N_m(X) = |X \cap U_\varepsilon(m, m+1, a, X)| \leq (2m+3)^d - (2m-1)^d < d 5^d m^{d-1} \quad (3.1)$$

holds.

*Proof.* Take  $x_j \in X \cap U_\varepsilon(m, m+1, a, X)$  and consider the balls  $B_{\varepsilon/2}(x_j) = \{x \in \mathcal{L} : |x - x_j| < \varepsilon/2\}$ , centered at  $x_j$ . They are contained in the spherical layer  $U_\varepsilon(m-1/2, m+3/2, a, X)$ , and pairwise disjoint. Since the volume of this layer is

$$|U_\varepsilon(m-1/2, m+3/2, a, X)| = \kappa_d \left[ ((m+3/2)\varepsilon)^d - ((m-1/2)\varepsilon)^d \right], \quad \kappa_d = \frac{\pi^{d/2}}{\Gamma(\frac{d}{2} + 1)}$$

is the volume of the unit ball in  $\mathbb{R}^d$ , and  $|B_{\varepsilon/2}(x_j)| = \kappa_d(\varepsilon/2)^d$ , the number  $N_m(X)$  satisfies (3.1), as claimed.  $\square$

As far as the Schoenberg symbol  $f$  in the definition of Schoenberg's matrices goes, we assume here that it is a nonnegative, monotone decreasing function on  $\mathbb{R}_+$ , and  $f(0) = 1$ , i.e.  $f \in \mathcal{M}_+$  (cf. (1.8)). Further assumptions on the behavior of  $f$  at infinity will vary.

We proceed with a simple technical result.

**Lemma 3.2.** *Let  $h \in \mathcal{M}_+$  and  $d \in \mathbb{N}$ . Then*

$$\sum_{m=1}^{\infty} m^{d-1} h(m) < \infty \iff \int_0^{\infty} t^{d-1} h(t) dt < \infty. \quad (3.2)$$

More precisely, for all  $p \in \mathbb{N}$

$$2^{-d+1} \int_p^{\infty} t^{d-1} h(t) dt \leq \sum_{m=p}^{\infty} m^{d-1} h(m) \leq d \int_{p-1}^{\infty} t^{d-1} h(t) dt. \quad (3.3)$$

*Proof.* An elementary inequality

$$\frac{m^{d-1}}{d} \leq \frac{m^d - (m-1)^d}{d} \leq m^{d-1}, \quad m \in \mathbb{N},$$

gives for  $h \in \mathcal{M}_+$

$$\int_{m-1}^m t^{d-1} h(t) dt \geq h(m) \int_{m-1}^m t^{d-1} dt = h(m) \frac{m^d - (m-1)^d}{d} \geq \frac{m^{d-1} h(m)}{d},$$

so summation over  $m$  leads to the right inequality in (3.3). Similarly,

$$\int_m^{m+1} t^{d-1} h(t) dt \leq h(m) \int_m^{m+1} t^{d-1} dt = h(m) \frac{(m+1)^d - m^d}{d} \leq (m+1)^{d-1} h(m),$$

and hence

$$\sum_{m=p}^{\infty} (m+1)^{d-1} h(m) \geq \int_p^{\infty} t^{d-1} h(t) dt.$$

It remains only to note that  $m+1 \leq 2m$  for  $m \in \mathbb{N}$ . □

For a one dimensional  $X$ , i.e.,  $d(X) = 1$ , condition (3.2) is just  $f \in L^1(\mathbb{R}_+)$ .

Recall that we write  $X \in \mathcal{X}_d$ ,  $d \leq n$ , if  $X \in \mathcal{X}_n$  and  $\dim \mathcal{L}(X) = d$ .

The following notion will be crucial in the second part of Theorem 3.4 below.

**Definition 3.3.** A set  $Y = \{y_j\}_{j \in \mathbb{N}} \in \mathcal{X}_d$  is called  $\delta$ -regular if there are constants  $c_0 = c_0(d, \delta, Y) > 0$  and  $r_0 = r_0(d, Y) \geq 0$ , independent from  $j$  such that

$$|Y_r^{(j)}(\delta)| \geq c_0(d, \delta, Y) r^{d-1}, \quad Y_r^{(j)}(\delta) := \{y_k \in Y : r \leq |y_k - y_j| < r + \delta\}, \quad (3.4)$$

for  $r \geq r_0$  and  $j \in \mathbb{N}$ .

For instance, the lattice  $\mathbb{Z}^n$  and its part  $\mathbb{Z}_+^n$  are  $\delta$ -regular for all  $\delta > 0$ . On the other hand, if  $X = \{x_k\}_{k \in \mathbb{N}} \in \mathbb{R}^n$ ,  $\mathcal{L}(X) = \mathbb{R}^n$  but  $X^{(p)} := \{x_k\}_{k \geq p} \subset \mathbb{R}^{n-1}$  then  $X$  is certainly irregular.

Note that for any regular set  $Y$  the number  $N_r^{(j)}$  of points in the set  $Y \cap \{y : |y - y_j| \leq r\}$  is subject to the bounds

$$c_1 r^d \leq N_r^{(j)} \leq c_2 r^d \quad (3.5)$$

for all large enough  $r$ . Here and in the proof of Theorem 3.4  $c_k$  stand for different positive constants which depend on  $d, \delta$ , and  $Y$ .

**Theorem 3.4** (=Theorem 1.4). *Let  $f \in \mathcal{M}_+$ ,  $X \in \mathcal{X}_n$  and let  $d = \dim \mathcal{L}(X)$ .*

(i) *If  $t^{d-1} f(\cdot) \in L^1(\mathbb{R}_+)$ , then the Schoenberg matrix  $\mathcal{S}_X(f)$  is bounded on  $\ell^2$  and*

$$\|\mathcal{S}_X(f)\| \leq 1 + d^2 \left( \frac{5}{d_*(X)} \right)^d \int_0^\infty t^{d-1} f(t) dt. \quad (3.6)$$

(ii) *Moreover,  $\mathcal{S}_X(f)$  has a bounded inverse whenever, in addition,*

$$d_*(X) > 5d^{2/d} \|t^{d-1} f\|_{L^1(\mathbb{R}_+)}^{1/d}. \quad (3.7)$$

(iii) *Conversely, let  $\mathcal{S}_Y(f)$  be bounded for at least one  $\delta$ -regular set  $Y$ . Then  $t^{d-1} f(\cdot) \in L^1(\mathbb{R}_+)$ .*

*Proof.* (i). We apply the Schur test to  $\mathcal{S}_X(f) = \|f(|x_k - x_j|)\|_{k,j \in \mathbb{N}}$ . For a fixed  $j \in \mathbb{N}$  and  $\varepsilon = d_*(X) > 0$  denote

$$X_m^{(j)} := \{x_k \in X : m\varepsilon \leq |x_k - x_j| < (m+1)\varepsilon\}, \quad m \in \mathbb{N}, \quad X_0^{(j)} = \{x_j\}. \quad (3.8)$$

By Lemma 3.1  $|X_m^{(j)}| < d 5^d m^{d-1}$ . Combining this estimate with the monotonicity of  $f$  yields

$$\begin{aligned} \sum_{k=1}^{\infty} f(|x_k - x_j|) &= 1 + \sum_{m=1}^{\infty} \sum_{x_k \in X_m^{(j)}} f(|x_k - x_j|) \leq 1 + \sum_{m=1}^{\infty} |X_m^{(j)}| f(m\varepsilon) \\ &\leq 1 + d 5^d \sum_{m=1}^{\infty} m^{d-1} f(m\varepsilon). \end{aligned} \quad (3.9)$$

The result now follows from the Schur test and Lemma 3.2 with  $h(\cdot) = f(\varepsilon \cdot)$ .

(ii). Going over to the second statement, one has as above

$$\sum_{k=1}^{\infty} |f(|x_k - x_j|) - \delta_{kj}| = \sum_{k \neq j} f(|x_k - x_j|) \leq d^2 \left( \frac{5}{d_*(X)} \right)^d \int_0^{\infty} t^{d-1} f(t) dt,$$

so  $\|S_X(f) - I\| < 1$  as soon as (3.7) holds and  $S_X(f)$  is invertible.

(iii). With no loss of generality assume that  $\mathcal{L}(X) = \mathbb{R}^d$ . At this point we make use of a particular labeling of the set  $X$  (generally speaking the way of enumeration of  $X$  makes no difference in our setting). Precisely, we label  $X$  by increasing of the distance from the origin

$$0 = |x_1| < |x_2| \leq |x_3| \leq \dots$$

For a ball  $B_r = B_r^d$  of radius  $r > 0$  centered at the origin we put  $E_r := X \cap B_r$  and  $N_r := |E_r|$ . Given  $x_j \in X$ , denote by  $p(j)$  the number of layers  $X_m^{(j)}$  which are contained in  $B_r$ . It is clear that for any  $x_j \in E_{r/2}$  one has  $p(j) \geq \lceil r/2\varepsilon \rceil$ .

From the Definition 3.3 and  $f \in \mathcal{M}_+$  we see that

$$\begin{aligned} \sum_{k=1}^{N_r} f(|x_k - x_j|) &\geq \sum_{m=1}^{p(j)} \sum_{x_k \in X_m^{(j)}} f(|x_k - x_j|) \geq c_3 \sum_{m=1}^{p(j)} m^{d-1} f(\varepsilon(m+1)) \\ &\geq c_4 \sum_{m=2}^{p(j)+1} m^{d-1} f(\varepsilon m). \end{aligned} \quad (3.10)$$

Since  $S_X(f)$  is bounded then on the test vector  $h = h_{N_r} = \frac{1}{\sqrt{N_r}}(1, 1, \dots, 1, 0, 0, \dots)$ ,  $\|h\| = 1$ , we have in view of (3.10) and (3.5) (with  $j = 1$ ,  $x_1 = 0$ )

$$\begin{aligned} \|S_X(f)\| &\geq |\langle S_X(f)h, h \rangle| = \frac{1}{N_r} \sum_{j=1}^{N_r} \sum_{k=1}^{N_r} f(|x_k - x_j|) \geq \frac{1}{N_r} \sum_{|x_j| < R/2} \sum_{k=1}^{N_r} f(|x_k - x_j|) \\ &\geq \frac{c_5}{N_r} N_{r/2} \sum_{m=2}^{\lceil r/2\varepsilon \rceil} m^{d-1} f(\varepsilon m) \geq c_6 \sum_{m=2}^{\lceil r/2\varepsilon \rceil} m^{d-1} f(\varepsilon m). \end{aligned}$$

Since  $r$  is arbitrarily large, the result follows from Lemma 3.2.  $\square$

*Remark 3.5.* The statement (iii) of the above theorem is particularly simple for  $d = 1$ .

A one-dimensional sequence  $X(\lambda) = \{x_k\}$ ,  $x_k = \lambda_k e$ , is called a Toeplitz-like sequence if

$$0 = \lambda_1 < \lambda_2 < \dots, \quad 0 < d_*(X) \leq \lambda_{i+1} - \lambda_i \leq d^*(X) < \infty, \quad (3.11)$$

for all  $i \in \mathbb{N}$ .

Assume now that the Schoenberg operator  $S_X(f)$  is bounded. Take the same test vector  $h_N = \frac{1}{\sqrt{N}}(1, 1, \dots, 1, 0, 0, \dots)$ ,  $\|h_N\| = 1$  and write

$$\|S_X(f)\| \geq \langle S_X(f)h_N, h_N \rangle = \frac{1}{N} \sum_{i,j=1}^N f(|x_i - x_j|) = f(0) + \frac{2}{N} \sum_{k=1}^{N-1} \sum_{i=1}^{N-k} f(\lambda_{i+k} - \lambda_i),$$

By (3.11),  $kd_*(X) \leq \lambda_{i+k} - \lambda_i \leq kd^*(X)$ , and in view of monotonicity

$$\|S\| \geq 2 \sum_{k=1}^{m-1} \left(1 - \frac{k}{m}\right) f(kd^*(X)) \geq 2 \sum_{k=1}^{m/2} \left(1 - \frac{k}{m}\right) f(kd^*(X)) \geq \sum_{k=1}^{m/2} f(kd^*(X)).$$

Thereby the series  $\sum_k f(kd^*(X))$  converges and Lemma 3.2 gives  $f \in L^1(\mathbb{R}_+)$ .

For  $\alpha$ -stable functions we have a simple condition for the boundedness of  $S_X(f)$  in terms of the Schoenberg measure  $\sigma$  (2.6).

**Corollary 3.6.** *Let  $f \in \Phi_\infty(\alpha)$ ,  $0 < \alpha \leq 2$ ,  $d \in \mathbb{N}$ , and let  $\sigma = \sigma_f$  be the Schoenberg measure in (2.6). Then*

$$\int_0^\infty t^{d-1} f(t) dt < \infty \iff \int_0^\infty s^{-\frac{d}{\alpha}} \sigma(ds) < \infty. \quad (3.12)$$

*In particular, the Schoenberg operator  $S_X(f)$  is bounded for all  $X \in \mathcal{X}_d$ , provided that the measure  $\sigma$  satisfies (3.12).*

*Proof.* It is clear that  $f \in \mathcal{M}_+$ . Next,

$$\begin{aligned} \int_0^\infty t^{d-1} f(t) dt &= \int_0^\infty t^{d-1} dt \int_0^\infty e^{-st^\alpha} \sigma(ds) = \int_0^\infty \sigma(ds) \int_0^\infty t^{d-1} e^{-st^\alpha} dt \\ &= \frac{1}{\alpha} \Gamma\left(\frac{d}{\alpha}\right) \int_0^\infty s^{-\frac{d}{\alpha}} \sigma(ds) < \infty. \end{aligned} \quad (3.13)$$

Theorem 3.4 completes the proof.  $\square$

Note that the above argument goes through for an arbitrary  $\alpha > 0$ .

We prove later in Theorem 4.7 that each Schoenberg operator  $S_X(f)$  with the symbol as in Corollary 3.6 is actually invertible.

As another direct consequence of Theorem 3.4 we have

**Corollary 3.7.** *Let  $f, g \in \mathcal{M}_+$  and  $f(t) = g(t)$  for  $t \geq t_0$ . If  $S_Y(f)$  is bounded for at least one regular set  $Y \in \mathcal{X}_d$ , then  $S_X(g)$  are bounded for all  $X \in \mathcal{X}_d$ .*

The monotonicity condition in (1.8) is somewhat restrictive. It is not at all necessary for Schoenberg's operator to be bounded.

**Proposition 3.8.** *Let  $f$  and  $h$  be real-valued functions on  $\mathbb{R}_+$ . Assume that  $|f| \leq h$  and the operator  $S_X(h)$  is bounded. Then so is  $S_X(f)$ . In particular, let  $f$  be a bounded function on  $\mathbb{R}_+$ , which is monotone decreasing for  $t \geq t_0(f)$  and  $t^{d-1}f(\cdot) \in L^1(\mathbb{R}_+)$ . Then  $S_X(f)$  is bounded.*

*Proof.* The Schoenberg matrix  $\mathcal{S}_X(h)$  dominates  $\mathcal{S}_X(f)$ , i.e.,  $h(|x_j - x_k|) \geq |f(|x_j - x_k|)|$ . Hence if  $\mathcal{S}_X(h)$  is bounded then by [3, Theorem 29.2], so is  $\mathcal{S}_X(f)$ .

Concerning the second statement, it is clear that  $f \geq 0$  on  $[t_0(f), \infty)$ . Put  $h(t) := \sup_{t \geq s} |f(s)|$ . Then  $h$  is a nonnegative function, monotone decreasing on  $\mathbb{R}_+$ ,  $h(0) > 0$  (we assume  $f \not\equiv 0$ ), and  $h = f$  on  $[t_0(f), \infty)$ , so (3.2) holds for  $h$ . By Theorem 3.4,  $\mathcal{S}_X(h)$  is bounded and as  $h \geq |f|$  on  $\mathbb{R}_+$ , then by the first part of the proof, so is  $\mathcal{S}_X(f)$ , as needed.  $\square$

**Corollary 3.9.** *Let  $g \in \Phi_n$ ,  $\alpha > 0$ , and  $e_\alpha(t) := e^{-\alpha t}$ . Then  $f_\alpha := e_\alpha g \in \Phi_n$  and for any  $d \in \mathbb{N}$  and any  $X \in \mathcal{X}_d$  the Schoenberg operator  $\mathcal{S}_X(f_\alpha)$  is bounded.*

*Proof.* Since  $e_\alpha \in CM_0(\mathbb{R}_+) \subset \Phi_\infty$ , then for any finite  $X$  the Schoenberg matrix  $\mathcal{S}_X(f_\alpha) = \mathcal{S}_X(e_\alpha) \circ \mathcal{S}_X(g)$ , being the Schur product of two non-negative matrices  $\mathcal{S}_X(e_\alpha)$  and  $\mathcal{S}_X(g)$ , is also non-negative. This proves the inclusion  $f_\alpha \in \Phi_n$ .

Next, since  $|f_\alpha(t)| \leq M e^{-\alpha t}$  with  $M = \|g\|_{C(\mathbb{R}_+)}$ , then  $t^{d-1} f_\alpha(\cdot) \in L^1(\mathbb{R}_+)$  with an arbitrary  $d \in \mathbb{N}$ . It remains to apply Proposition 3.8.  $\square$

## 3.2 Fredholm property

We discuss here the situation when  $\mathcal{S}_X(f)$  is a Fredholm operator, more precisely,

$$\mathcal{S}_X(f) = I + T, \quad T \in \mathfrak{S}_\infty(\ell^2) \quad (3.14)$$

is a compact operator on  $\ell^2$ . In this case one should impose a much stronger condition on  $X$  than just  $d_*(X) > 0$ .

**Theorem 3.10.** *Let  $X = \{x_k\}_{k \in \mathbb{N}} \subset \mathbb{R}^d$  satisfy*

$$\lim_{\substack{i, j \rightarrow \infty \\ i \neq j}} |x_i - x_j| = +\infty. \quad (3.15)$$

*Let  $f \in \mathcal{M}_+$  with  $t^{d-1}f \in L^1(\mathbb{R}_+)$ . Then (3.14) holds. In particular,  $\mathcal{S}_X(f)$  has bounded inverse whenever  $\ker \mathcal{S}_X(f) = \{0\}$ .*

*Conversely, let  $f$  be a strictly positive, monotone decreasing function on  $\mathbb{R}_+$ ,  $f(0) = 1$ , and  $t^{d-1}f \in L^1(\mathbb{R}_+)$ . Then (3.14) implies (3.15).*

*Proof.* To apply Lemma 2.5 we argue as in the proof of the Theorem 3.4. According to Lemma 3.1 for each  $p \in \mathbb{N}$  there is  $q = q(p) \in \mathbb{N}$  so that for  $j \geq p$

$$\begin{aligned} \sum_{k=p}^{\infty} |f(|x_k - x_j|) - \delta_{kj}| &= \sum_{k \geq p, k \neq j} f(|x_k - x_j|) = \sum_{m=q}^{\infty} \sum_{x_k \in X_m^{(j)}} f(|x_k - x_j|) \leq d 5^d \sum_{m=q}^{\infty} m^{d-1} f(d_*(X)m) \\ &\leq d^2 \left( \frac{5}{d_*(X)} \right)^d \int_{d_*(q-1)}^{\infty} t^{d-1} f(t) dt. \end{aligned}$$

Condition (3.15) implies  $q(p) \rightarrow \infty$  as  $p \rightarrow \infty$  and so operator  $T = \mathcal{S}_X(f) - I$  is compact by Lemma 2.5.

Conversely, suppose that there are two sequences  $\{i_m\}, \{j_m\}$  so that  $i_m \neq j_m$ , both tend to infinity as  $m \rightarrow \infty$  and  $\sup_m |x_{i_m} - x_{j_m}| \leq C < \infty$ . Then

$$0 < f(C) \leq f(|x_{i_m} - x_{j_m}|) = \langle \mathcal{S}_X(f) e_{j_m}, e_{i_m} \rangle = \langle T e_{j_m}, e_{i_m} \rangle,$$

which contradicts the compactness of  $T$ . The proof is complete.  $\square$

*Example 3.11.* We show that in the converse statement of Theorem 3.10 the condition  $f > 0$  cannot be relaxed to  $f \geq 0$ . Take the truncated power function

$$f(t) = (1 - t)_+^l, \quad l > 0.$$

It is known [15, 35] that  $f \in \Phi_n$  if and only if  $l \geq \frac{n+1}{2}$ . As a test sequence  $X = \{x_k\}_{k \in \mathbb{N}}$  we put  $x_k = a_k \xi$ ,  $\|\xi\| = 1$ , with

$$a_1 = 0, \quad a_2 = \frac{1}{2}, \quad a_k = k, \quad k = 3, 4, \dots,$$

so that  $f(|x_2 - x_1|) = 2^{-l}$ ,  $f(|x_i - x_j|) = 0$  for the rest of the pairs  $j \neq i$ . The Schoenberg operator now takes the form

$$S_X(f) = \begin{bmatrix} A & \\ & I \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 2^{-l} \\ 2^{-l} & 1 \end{bmatrix}$$

and  $I$  is a unit matrix. It is clear that  $S_X(f) = I + T$ ,  $rk T = 2$ , but (3.15) is false.

### 3.3 Unbounded Schoenberg operators

Conditions on an infinite matrix  $\mathcal{A}$  for the corresponding linear operator  $A$  on  $\ell^2$  to be bounded are rather stringent. These conditions fail to hold for a number of Schoenberg's matrices (cf. Example 3.27).

To broaden the area of our study, consider an infinite Hermitian matrix  $\mathcal{A} = \|a_{kj}\|_{k,j \in \mathbb{N}}$ ,  $a_{jk} = \bar{a}_{kj}$ , satisfying the following conditions

$$\sum_{k=1}^{\infty} |a_{kj}|^2 < \infty, \quad \forall j \in \mathbb{N}. \quad (3.16)$$

Such matrix defines in a natural way a linear operator  $A'$  on  $\ell^2$  which act on the standard basis vectors  $\{e_k\}_{k \in \mathbb{N}}$ ,  $(e_k)_m = \delta_{km}$ , as

$$A'e_j = \sum_{k=1}^{\infty} a_{kj} e_k, \quad j \in \mathbb{N},$$

extended by linearity to the linear span  $\mathcal{L}$  of  $\{e_k\}_{k \in \mathbb{N}}$ , so  $A'$  is densely defined and  $\text{dom}(A') \supseteq \mathcal{L}$ . Being symmetric (since  $\mathcal{A}$  is a Hermitian matrix),  $A'$  is closable, and we denote by  $A = \overline{A'}$  its closure. The operator  $A$  is called a minimal operator associated with  $\mathcal{A}$ .

Matrices (3.16) are usually referred to as *unbounded Hermitian matrices* (unless  $A$  is a bounded operator).

A maximal operator associated with such matrix  $\mathcal{A}$  is given by

$$A_{\max} f = \sum_{k=1}^{\infty} b_k e_k, \quad b_k = \sum_{j=1}^{\infty} a_{kj} x_j, \quad (3.17)$$

on the domain

$$\text{dom}(A_{\max}) = \left\{ f = \sum_{k=1}^{\infty} x_k e_k : \sum_{k=1}^{\infty} \left| \sum_{j=1}^{\infty} a_{kj} x_j \right|^2 < \infty \right\}.$$

It is known (see, e.g., [3, Theorem 53.2]) that  $A_{\max} = A^*$ .

Conversely, given a closed symmetric operator  $A$  on a Hilbert space  $\mathcal{H}$ , an orthonormal basis  $\{h_k\}_{k \in \mathbb{N}}$  is called a basis of the matrix representation of  $A$  if



- $h_k \in \text{dom}(A)$ ,  $k \in \mathbb{N}$ ;
- $A$  is a minimal closed operator sending  $h_k$  to  $Ah_k$ ,  $k \in \mathbb{N}$ .

A curious property of certain Schoenberg's matrices is that the validity of (3.16) for at least one value of  $j$  implies relation (3.16) to hold for all  $j \in \mathbb{N}$ . We begin with the technical lemma.

Let us say that a finite positive Borel measure  $\sigma$  on  $\mathbb{R}_+$  possesses a doubling property if there is  $\kappa > 0$  so that

$$\sigma[2u, 2v] \leq \kappa \sigma[u, v], \quad \forall [u, v] \subset \mathbb{R}_+. \quad (3.18)$$

**Lemma 3.12.** *Let  $f \in CM_0(\mathbb{R}_+)$  and  $\xi, \eta \in \mathbb{R}^n$ . Then there is a constant  $C = C(f, \xi, \eta) > 0$  such that*

$$f(|x - \xi|) < Cf(|x - \eta|), \quad \forall x \in \mathbb{R}^n. \quad (3.19)$$

*The same conclusion is true for  $f \in \Phi_\infty = \Phi_\infty(2)$  as long as its Schoenberg measure  $\sigma$  (2.6) possesses the doubling property.*

*Proof.* First, let  $f \in CM_0(\mathbb{R}_+)$ . Choose  $a = a_f > 0$  so that

$$\int_0^a \tau(ds) > \frac{1}{2} \Rightarrow \int_a^\infty \tau(ds) < \frac{1}{2}. \quad (3.20)$$

We show that (3.19) actually holds with  $C = 2e^{a|\xi - \eta|}$ . Consider two cases.

1. Let first  $|x - \eta| \leq |\xi - \eta|$ . Then since  $f \leq 1$ , one has

$$f(|x - \eta|) = \int_0^\infty e^{-s|x - \eta|} \tau(ds) \geq \int_0^a e^{-s|x - \eta|} \tau(ds) > \frac{1}{2} e^{-a|x - \eta|} \geq \frac{1}{2} e^{-a|\xi - \eta|} f(|x - \xi|),$$

as needed.

2. Let now  $|x - \eta| > |\xi - \eta|$ , so  $|x - \xi| \geq |x - \eta| - |\xi - \eta| > 0$ . The function  $f$  is certainly monotone decreasing, so

$$\begin{aligned} f(|x - \xi|) &\leq f(|x - \eta| - |\xi - \eta|) = \int_0^\infty \exp(-s|x - \eta| + s|\xi - \eta|) \tau(ds) \\ &= \left\{ \int_0^a + \int_a^\infty \right\} \exp(-s|x - \eta| + s|\xi - \eta|) \tau(ds) = I_1 + I_2. \end{aligned}$$

Obviously, for every nonnegative and monotone decreasing function  $u$  on  $\mathbb{R}_+$ , condition (3.20) implies

$$\int_0^a u(s) \tau(ds) \geq \frac{u(a)}{2} > u(a) \int_a^\infty \tau(ds) \geq \int_a^\infty u(s) \tau(ds).$$

Hence  $I_2 \leq I_1$ . To bound  $I_1$  note that

$$I_1 \leq e^{a|\xi - \eta|} \int_a^\infty e^{-s|x - \eta|} \tau(ds) = e^{a|\xi - \eta|} f(|x - \eta|),$$

and (3.19) follows.

Concerning functions  $f \in \Phi_\infty$ , the reasoning is identical (with the obvious replacement of  $\tau$  with  $\sigma$ ) up to the bound of  $I_1$ , where the doubling property comes into play. We now have

$$I_1 = \int_0^a \exp(-s(|x - \eta| - |\xi - \eta|)^2) \sigma(ds) \leq e^{a|\xi - \eta|^2} \int_0^a e^{-\frac{s}{2}|x - \eta|^2} \sigma(ds) \leq e^{a|\xi - \eta|^2} f\left(\frac{|x - \eta|}{\sqrt{2}}\right).$$

It remains only to note that

$$f\left(\frac{r}{\sqrt{2}}\right) = \int_0^\infty e^{-\frac{s}{2}r^2} \sigma(ds) \leq \kappa \int_0^\infty e^{-sr^2} \sigma(ds) = \kappa f(r), \quad r > 0$$

because of the doubling property (3.18). The proof is complete.  $\square$

**Proposition 3.13.** *Let  $f \in CM_0(\mathbb{R}_+)$ ,  $X = \{x_k\}_{k \in \mathbb{N}} \subset \mathbb{R}^n$ , and let  $\mathcal{S}_X(f)$  be the corresponding Schoenberg matrix. If at least one column of  $\mathcal{S}$  belongs to  $\ell^2$ , then (3.16) holds and  $\{e_k\}_{k \in \mathbb{N}}$  is a basis of the matrix representation for the minimal operator  $A$  associated with  $\mathcal{S}_X(f)$ . The same conclusion is true for  $f \in \Phi_\infty$  as long as its Schoenberg measure  $\sigma$  possesses the doubling property.*

*Proof.* Let the first column of  $\mathcal{S}$  belong to  $\ell^2$ . By Lemma 3.12 one has

$$\sum_{j=1}^\infty |f(x_j - x_k)|^2 \leq C^2 \sum_{j=1}^\infty |f(x_j - x_1)|^2 < \infty \quad (3.21)$$

for each  $k = 2, 3, \dots$ . The statement about the basis of the matrix representation is obvious.  $\square$

*Remark 3.14.* It is easy to see that in general for functions off  $CM_0(\mathbb{R}_+)$  the doubling property (3.18) for  $\sigma$  cannot be dropped.

Put

$$a_n := \sqrt{\log n + 2 \log \log n}, \quad n \geq 2.$$

Then clearly

$$\sum_{n=2}^\infty e^{-a_n^2} = \sum_{n=2}^\infty \frac{1}{n \log^2 n} < \infty, \quad \sum_{n=2}^\infty e^{-(a_n-1)^2} = \frac{1}{e} \sum_{n=2}^\infty \frac{e^{2a_n}}{n \log^2 n} = \infty.$$

Consider now the Schoenberg matrix  $\mathcal{S}_X(f)$  with

$$f(t) = e^{-t^2} \in \Phi_\infty \setminus CM_0(\mathbb{R}_+), \quad X = \{x_k\}_{k \in \mathbb{N}} \subset \mathbb{R}^1: \quad x_1 = 0, \quad x_2 = 1, \quad x_n = a_n, \quad n \geq 3.$$

Then

$$\sum_{n=1}^\infty f^2(|x_n - x_1|) < \infty, \quad \sum_{n=1}^\infty f^2(|x_n - x_2|) = \infty.$$

Certainly, now  $\sigma = \delta\{1\}$  has no doubling property. Note that in this instance the conclusion of Lemma 3.12 is false either.

In the above example the set  $X$  is not separated, that is,  $d_*(X) = 0$ . As we will see later in Theorem 4.7, the Schoenberg operator  $S_X(e^{-t^2})$  is bounded and invertible whenever  $d_*(X) > 0$ , so all columns belong to  $\ell^2$ .

There is an intermediate condition on the Schoenberg matrix  $\mathcal{S}_X(f)$  between (3.16) and the boundedness. Precisely,

$$\sup_j \sum_{k=1}^\infty f^2(|x_k - x_j|) = C(f, X) < \infty. \quad (3.22)$$

In other words,  $\sup_j \|S_X(f)e_j\| < \infty$ .

Recall that  $\delta$ -regular sets are defined in Definition 3.3 above.

**Proposition 3.15.** *Let  $f \in \mathcal{M}_+$ , that is,  $f$  enjoys condition (1.8), and*

$$\int_0^\infty t^{d-1} f^2(t) dt < \infty. \quad (3.23)$$

*Then (3.22) holds for each separated set  $X \in \mathcal{X}_d$ . Conversely, assume that*

$$\sum_{k=1}^\infty f^2(|y_k - y_j|) = C(f, Y) < \infty \quad (3.24)$$

*for some  $j \in \mathbb{N}$  and at least one  $\delta$ -regular set  $Y$ . Then (3.23) holds with  $d = \dim \mathcal{L}(Y)$ .*

*Proof.* Let (3.23) hold. We apply Lemma 3.2 with  $h = f^2$  and obtain as above

$$\sum_{k=1}^\infty f^2(|x_k - x_j|) \leq 1 + \sum_{m=1}^\infty |X_m^{(j)}| f^2(d_*(X)m) \leq 1 + d^2 \left( \frac{5}{d_*(X)} \right)^d \int_0^\infty s^{d-1} f^2(s) ds,$$

so (3.22) follows.

Conversely, let  $f$  satisfy (3.24) for some  $j \in \mathbb{N}$  and some  $\delta$ -regular set  $Y = \{y_j\}_{j \in \mathbb{N}}$ . In view of the lower bound (3.3) one has by Lemma 3.2,

$$\begin{aligned} \sum_{k=1}^\infty f^2(|y_k - y_j|) &= 1 + \sum_{m=1}^\infty \sum_{y_k \in Y_m^{(j)}} f^2(|y_k - y_j|) \geq 1 + c_2(d) \sum_{m=1}^\infty m^{d-1} f^2(d_*(Y)(m+1)) \\ &\geq 1 + c_3(d) \sum_{m=2}^\infty m^{d-1} f^2(d_*(Y)m) \geq 1 + c_4(d) \int_{2d_*(Y)}^\infty s^{d-1} f^2(s) ds. \end{aligned}$$

The proof is complete.  $\square$

**Corollary 3.16.** *If  $e_j \in \text{dom } S_X(f)$  for some  $j \in \mathbb{N}$  and all  $X \in \mathcal{X}_n$  then (3.22) holds.*

*Remark 3.17.* Condition (3.22) for an individual set  $X$  has nothing to do with bound (3.23). Indeed, let  $f$  tend to zero arbitrarily slowly as  $x \rightarrow \infty$ . Choose a sequence of positive numbers  $\{t_k\}$ ,  $t_1 = 0$  so that  $f(t_k) \leq e^{-k}$ . Now take a set  $X = \{x_k\}_{k \in \mathbb{N}}$  with  $x_k = t_k \xi$ ,  $k \in \mathbb{N}$ ,  $\xi$  a unit vector. Then

$$\sum_{i=1}^\infty f^2(|x_k - x_1|) \leq \sum_k e^{-2k} < \infty$$

regardless of whether condition (3.23) holds or not.

The example below illustrates Proposition 3.15 and Theorem 3.4.

*Example 3.18.* Let  $h(r) = (1+r)^{-1} \in CM_0(\mathbb{R}_+)$ . Take  $X = \mathbb{Z}_+^2 = \{(p, q) : p, q \in \mathbb{Z}_+\}$  labeled in the following way

$$X = \bigcup_{m=0}^\infty X_m, \quad X_m = \{x_k^{(m)}\}_{k=0}^m, \quad x_k^{(m)} = (m-k, k), \quad X_0 = \{(0, 0)\}.$$

As  $|x_k^{(m)}|^2 = (m-k)^2 + k^2 = m^2 + 2k(k-m) \leq m^2$ , we can easily compute the sum in (3.22)

$$\sum_{m=0}^\infty \sum_{k=0}^m h^2(|x_k^{(m)}|) = \sum_{m=0}^\infty \sum_{k=0}^m \frac{1}{(1+|x_k^{(m)}|)^2} \geq \sum_{m=0}^\infty \sum_{k=0}^m \frac{1}{(1+m)^2} = \sum_{m=0}^\infty \frac{1}{1+m} = +\infty,$$

which is consistent with Proposition 3.15, since  $d = \dim \mathcal{L}(X) = 2$ , and condition (3.23) is violated.

On the other hand, let  $X = \mathbb{Z}_+$ , so we come to a version of the well-known Hilbert–Toeplitz matrix

$$\mathcal{S}_X(h) = \|(1 + |i - j|)^{-1}\|_{i,j \in \mathbb{N}}, \quad h(r) = \frac{1}{1+r} = \int_0^\infty e^{-sr} e^{-s} ds. \quad (3.25)$$

Now  $d = 1$ , so by Proposition 3.15, (3.24) holds. Yet the operator  $S_X(h)$  is unbounded in view of Theorem 3.4 ( $\mathbb{Z}_+$  is a 1-regular set). We show later in Proposition 3.26 that  $S_X(h)$  is a positive definite and self-adjoint operator.

An important property of a minimal Schoenberg operator  $A = S_X(f)$  constitutes the content of the following theorem.

**Theorem 3.19.** *Let  $f \in \Phi_\infty(\alpha)$ ,  $\alpha \in (0, 2]$ ,  $X = \{x_j\}_{j \in \mathbb{N}} \subset \mathbb{R}^n$ , and  $X \in \mathcal{X}_n$ . Assume that the Schoenberg matrix  $\mathcal{S}_X(f)$  satisfies condition (3.16). Then the (minimal) Schoenberg operator  $S_X(f)$  associated with the matrix  $\mathcal{S}_X(f)$  is a symmetric positive definite operator, i.e.,*

$$\langle S_X(f)\xi, \xi \rangle \geq \varepsilon \|\xi\|^2, \quad \xi \in \text{dom } S_X(f), \quad \varepsilon > 0, \quad (3.26)$$

and so the deficiency indices  $n_\pm(A) = \dim \ker(A^*)$ . In particular,  $S_X(f)$  is self-adjoint if and only if  $\ker A^* = \{0\}$ .

*Proof.* According to Theorem 4.7, the function  $f \in \Phi_\infty(\alpha)$  is strongly  $X$ -positive definite, i.e., there exists  $\varepsilon > 0$  such that

$$\sum_{j,k=1}^N f(|x_j - x_k|) \xi_j \bar{\xi}_k \geq \varepsilon \sum_{j=1}^N |\xi_j|^2, \quad \xi = \{\xi_j\}_1^N \in \mathbb{C}^N, \quad \forall N \in \mathbb{N}. \quad (3.27)$$

Due to assumption (3.16) the basis  $\{e_j\}_{j \in \mathbb{N}}$  is a basis of the matrix representation of the minimal operator  $S_X(f)$  associated with the Schoenberg matrix  $\mathcal{S}_X(f)$ . Therefore inequality (3.27) means that for any finite vector  $\xi = (\xi_1, \xi_2, \dots, \xi_N, 0, 0, \dots)$

$$(A\xi, \xi) = (A'\xi, \xi) \geq \varepsilon \sum_{k=1}^N |\xi_k|^2 = \varepsilon \|\xi\|^2.$$

Taking the closure we get the statement. □

Note that the proof of our main result about  $\Phi_\infty$ -functions– Theorem 4.7 – in the next section is completely independent of the above Theorem 3.19.

The converse to Theorem 3.19 is true in more general setting.

**Proposition 3.20.** *Assume that the Schoenberg matrix  $\mathcal{S}_X(f)$ ,  $f \in \Phi_n$ , satisfies condition (3.16), and the (minimal) Schoenberg operator  $S_X(f)$  associated with the matrix  $\mathcal{S}_X(f)$  satisfies (3.26), i.e., it is positive definite. Then  $X$  is separated, i.e.,  $d_*(X) > 0$ .*

*Proof.* In the above assumptions one has

$$\langle S_X(f)h, h \rangle \geq c \|h\|^2, \quad 0 < c < \infty \quad (3.28)$$

for each  $h \in \text{dom}(S_X(f))$ . Hence putting  $h = e_k - e_j \in \text{dom}(S_X(f))$ , we see that

$$\langle S_X(f)h, h \rangle = 2f(0) - 2f(|x_k - x_j|) \geq 2c,$$

so  $f(|x_k - x_j|) \leq f(0) - c$ ,  $c > 0$ , which immediately implies  $d_*(X) > 0$ . □

Proposition 3.20 says that if  $d_*(X) = 0$  and  $S_X(f)$  is bounded for  $f \in \Phi_n$ , then  $0 \in \sigma(S_X(f))$ . It is easy to manufacture such  $X$  for  $f(t) = e^{-t}$  (cf. [17, Lemma 3.7]).

There is a simple function theoretic analogue of Proposition 3.20.

**Proposition 3.21.** *If  $f \in \Phi_n$  is strongly  $X$ -positive definite, then  $X$  is separated.*

*Proof.* By the definition we have for all  $k, j$

$$f(0)(|\xi_1|^2 + |\xi_2|^2) - f(|x_j - x_k|)(\xi_1 \bar{\xi}_2 + \bar{\xi}_1 \xi_2) \geq c(|\xi_1|^2 + |\xi_2|^2).$$

By putting  $\xi_1 = \xi_2 \neq 0$  we see that

$$f(0) - f(|x_j - x_k|) \geq c > 0,$$

so  $X \in \mathcal{X}_n$ , as needed. □

### 3.4 Schoenberg–Toeplitz operators

Although we have no sufficient conditions for general Schoenberg operators  $S_X(f)$  to be self-adjoint, Theorem 3.19 gives an essential step toward proving self-adjointness, since it reduces this problem to the study of  $\ker A^*$ .

**Definition 3.22.** (i) Let  $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$ . Recall that a matrix  $\mathcal{A} := \|a_{jk}\|_{j,k \in \mathbb{N}_0}$  is called a *Toeplitz matrix* if there is a numerical sequence  $\{a_m\}_{m \in \mathbb{Z}}$  such that  $a_{jk} = a_{j-k}$  for every  $j, k \in \mathbb{N}_0$ .

(ii) An operator  $A$  defined at least on the set of analytic polynomials  $Pol_+$  is called a Toeplitz operator if its matrix with respect to the basis  $\{z^k\}_{k \in \mathbb{N}_0} = \{e^{ik\varphi}\}_{k \in \mathbb{N}_0}$  is the Toeplitz matrix.

It is known that a Toeplitz operator is characterized by the identity

$$S^*AS = A, \tag{3.29}$$

where  $S$  is a unilateral shift in  $l^2$ . According to the basic assumption (3.16) the Toeplitz matrix  $\mathcal{A}$  defines an operator in  $\ell^2$  if  $\{a_k\} \in l^2(\mathbb{Z})$ , i.e.,

$$\sum_{j \in \mathbb{Z}} |a_j|^2 < \infty. \tag{3.30}$$

In this case the *Toeplitz symbol* is a function given by

$$a(A, e^{i\varphi}) := \sum_{k \in \mathbb{Z}} a_k e^{ik\varphi} \in L^2(\mathbb{T}), \quad \varphi \in [-\pi, \pi], \tag{3.31}$$

**Lemma 3.23.** *Let  $a_{-j} = \bar{a}_j$ ,  $j \in \mathbb{N}$ , i.e., the Toeplitz matrix  $\mathcal{A} = \|a_{j-k}\|_{j,k \in \mathbb{N}_0}$  is a Hermitian matrix. Assume also that  $\mathcal{A}$  satisfies (3.30) and the minimal symmetric Toeplitz operator  $A$  associated in  $l^2(\mathbb{N})$  with  $\mathcal{A}$  is semibounded from below. Then it is self-adjoint,  $A = A^*$ .*

*Proof.* Without loss of generality we can assume that  $A$  is positive definite. In this case it suffices to make sure that the conjugate (maximal) operator  $A^*$  has the trivial kernel. Since  $A^* = A_{max}$  acts by means of the same matrix  $\mathcal{S}_X(f)$  (but defined on the maximal domain), the latter property is equivalent to

$$\begin{bmatrix} a_0 & a_1 & a_2 & \dots \\ a_{-1} & a_0 & a_1 & \dots \\ a_{-2} & a_{-1} & a_0 & \dots \\ \vdots & \vdots & \vdots & \end{bmatrix} \begin{bmatrix} p_0 \\ p_1 \\ p_2 \\ \vdots \end{bmatrix} = \mathbb{O} \Rightarrow p_j \equiv 0, \quad p = \{p_j\} \in \ell^2. \tag{3.32}$$

To prove implication (3.32) it is instructive to rephrase the problem in the function theoretic terms.

Let  $U$  denote the multiplication (shift) operator on  $L^2(\mathbb{T})$ . The equality in (3.32) means that the function

$$p(t) := \sum_{j \geq 0} p_j t^j \in H^2$$

is orthogonal to the system  $\{U^k a\}_{k \geq 0}$ , where  $a \in L^2(\mathbb{T})$  is the Toeplitz symbol (3.31). In other words,  $p a \in L^1(\mathbb{T})$  is orthogonal to all powers  $\{t^k\}_{k \geq 0}$ , i.e.,  $p_- := p a \in H_-^1$ .

A positive definiteness of the minimal operator  $A$  reads as follows

$$(Aq, q) = \sum_{k, j=0}^N a_{k-j} q_j \bar{q}_k = \int_{\mathbb{T}} a(t) |q(t)|^2 m(dt) \geq \varepsilon \|q\|_{L^2(\mathbb{T})}^2, \quad q(t) := \sum_{j=0}^N q_j t^j, \quad \varepsilon > 0, \quad (3.33)$$

for an arbitrary analytic polynomial  $q$ ,  $m$  is the normalized Lebesgue measure on  $\mathbb{T}$ . It is clear from (3.33) that  $a(t) \geq \varepsilon$  for a.e.  $t = e^{i\varphi} \in \mathbb{T}$ . Therefore (see [10, Theorem II.4.6]) there is an outer function  $D$  such that

$$a(t) = |D(t)|^2, \quad D \in H^2, \quad D^{-1} \in H^\infty.$$

We have  $p(t) a(t) = p(t) |D(t)|^2 = p_-(t) \in H_-^1$  and hence

$$p(t) D(t) = \frac{p_-(t)}{\overline{D(t)}}.$$

But the left-hand side of the latter equality belongs to  $H^1$ , whereas the right-hand side lies in  $H_-^1$  which yields  $p \equiv 0$ , as claimed. The proof is complete.  $\square$

A sequence  $X = \{x_k\}_{k \in \mathbb{N}} \subset \mathbb{R}^n$  is called a *Toeplitz sequence*, if  $|x_i - x_j| = |i - j|$  for  $i, j \in \mathbb{N}$ . The latter is equivalent (recall that by our convention  $x_1 = 0$ ) to  $x_k = (k - 1)\xi$ ,  $\xi \in \mathbb{R}^n$ , and  $|\xi| = 1$ , so  $d = \dim \mathcal{L}(X) = 1$ . In this case  $S_X(f)$  is a Toeplitz operator, which will be called a *Schoenberg–Toeplitz operator*. The Toeplitz symbol  $a$  (3.31) takes now the form

$$a(f, e^{i\varphi}) := \sum_{k \in \mathbb{Z}} f(|k|) e^{ik\varphi}. \quad (3.34)$$

*Remark 3.24.* (i) Self-adjointness of not necessarily positive Toeplitz operators with the Toeplitz symbol from  $BMO(\mathbb{T})$  was established by V. Peller [21]. This is the case for the Hilbert–Toeplitz operator (3.25) with the Toeplitz symbol

$$a(h, t) = 1 - 2\Re \frac{\log(1 - t)}{t} \in BMO(\mathbb{T}),$$

but not for general Schoenberg–Toeplitz operators with Toeplitz symbols (3.42).

(ii) Semi-bounded Toeplitz operators have been studied in several papers (see [22] and references therein). For instance, it is proved in [23] that the Friedrichs extension  $A_F$  of  $A$  has absolutely continuous spectrum. However, according to Lemma 3.23,  $A_F = A$ .

In the rest of the section we will focus on the Schoenberg–Toeplitz operators  $S_X(f)$  with symbols  $f \in \Phi_\infty = \Phi_\infty(2)$ . We clarify and complete Corollary 3.6 for such operators and describe their spectra in terms of the Schoenberg measures  $\sigma = \sigma_f$ .

**Proposition 3.25.** *Let  $f \in \Phi_\infty$  and let  $\sigma$  be its Schoenberg measure (2.6). The Schoenberg–Toeplitz matrix  $\mathcal{S}_X(f)$  defines a minimal operator  $S_X(f)$  in  $\ell^2$  if and only if  $f \in L^2(\mathbb{R}_+)$ . In this case  $S_X(f)$  is self-adjoint, its spectrum is purely absolutely continuous and fills in the interval*

$$\sigma(S_X(f)) = \sigma_{ac}(S_X(f)) = [c_-, c_+],$$

$$0 < c_- := \int_0^\infty \vartheta_3(\pi, e^{-s}) \sigma(ds) < c_+ := \int_0^\infty \vartheta_3(0, e^{-s}) \sigma(ds) \leq +\infty, \quad (3.35)$$

where  $\vartheta_3$  is the Jacobi theta-function.

Moreover, the operator  $S_X(f)$  is bounded if and only if  $f \in L^1(\mathbb{R}_+)$ , or, equivalently,

$$\int_0^\infty \frac{\sigma(ds)}{\sqrt{s}} < \infty. \quad (3.36)$$

*Proof.* As the Schoenberg symbol  $f$  is a nonnegative and monotone decreasing function, conditions  $f \in L^2(\mathbb{R}_+)$  and  $\{f(k)\}_{k \geq 0} \in \ell^2$  are equivalent, so (3.30) is met. Next, for  $f \in \Phi_\infty$  the corresponding minimal operator is symmetric and strongly positive definite by Theorem 3.19. Hence  $S_X(f)$  is self-adjoint in view of Lemma 3.23.

Consider the kernel function  $e_s(u) := e^{-su^2}$ ,  $s > 0$ , so  $S_X(e_s) = \|e^{-s|i-j|^2}\|_{i,j \in \mathbb{N}}$ . Since  $e_s(\cdot) \in L^1(\mathbb{R}_+)$ , the operator  $S_X(e_s)$  is bounded by Theorem 3.4. The corresponding Toeplitz symbol is given by (3.34). It can now be expressed by means of the Jacobi theta-function

$$a(e_s, e^{i\varphi}) = \sum_{k \in \mathbb{Z}} e^{-s|k|^2} e^{ik\varphi} = \vartheta_3\left(\frac{\varphi}{2}, e^{-s}\right).$$

It is well known (see [32, Chapter 21]) that  $\vartheta_3$  is positive on the real line and

$$\frac{\vartheta_3'(\varphi)}{\vartheta_3(\varphi)} = -4 \sin 2\varphi \sum_{k=1}^{\infty} \frac{q^{2k-1}}{1 + 2q^{2k-1} \cos 2\varphi + q^{4k-2}}, \quad q = e^{-s},$$

so  $a(e_s)$  is monotone decreasing on  $[0, \pi]$  ( $a(e_s)$  is “bell-shaped” on  $[-\pi, \pi]$ ). By the Hartman–Wintner theorem (see, e.g., [19, Theorem 4.2.7]) its spectrum agrees with the range of  $a(f)$ , so it is the interval

$$\sigma(S_X(e_s)) = a(e_s, \mathbb{T}) = [a(e_s, -1), a(e_s, 1)] = [\vartheta_3\left(\frac{\pi}{2}, q\right), \vartheta_3(0, q)].$$

For a general function  $f \in \Phi_\infty$  the Toeplitz symbol  $a(f)$  of  $S_X(f) = \|f(|i-j|)\|_{i,j \in \mathbb{N}}$  can be computed as

$$a(f, e^{i\varphi}) = \sum_{k \in \mathbb{Z}} f(|k|) e^{ik\varphi} = \sum_{k \in \mathbb{Z}} e^{ik\varphi} \int_0^\infty e^{-s|k|^2} \sigma(ds) = \int_0^\infty \vartheta_3\left(\frac{\varphi}{2}, e^{-s}\right) \sigma(ds), \quad \varphi \neq 0. \quad (3.37)$$

It is easily seen that

$$\vartheta_3\left(\frac{\pi}{2}, e^{-s}\right) \leq \vartheta_3\left(\frac{\varphi}{2}, e^{-s}\right) \leq \vartheta_3(0, e^{-s}) = \sum_{k \in \mathbb{Z}} e^{-sk^2} \sim \frac{1}{\sqrt{s}}, \quad s \rightarrow +0, \quad 0 \leq \varphi \leq \pi. \quad (3.38)$$

Again by the Hartman–Wintner theorem, the spectrum of  $S_X(f)$  agrees with the range of  $a(f)$ , which is exactly the interval given by (3.35). Its absolute continuity is a standard fact in the theory of Toeplitz operators, (see, e.g., [22, p. 64]).

By Theorem 3.4 the boundedness of  $S_X(f)$  is equivalent to  $f \in L^1(\mathbb{R}_+)$ . In turn, the latter is equivalent to (3.36) by Corollary 3.6, applied with  $\alpha = 2$  and  $d = 1$ . The proof is complete.  $\square$



It is easy to express the inclusion  $f \in \Phi_\infty \cap L^2(\mathbb{R}_+)$  in terms of  $\sigma$  (cf. (3.36))

$$\int_{\mathbb{R}_+^2} \frac{\sigma(ds_1) \sigma(ds_2)}{\sqrt{s_1 + s_2}} < \infty. \quad (3.39)$$

Next, we provide a similar result for  $f \in CM_0(\mathbb{R}_+)$ .

**Proposition 3.26.** *Let  $f \in CM_0(\mathbb{R}_+)$ ,  $\tau$  be its Bernstein measure (2.5). The Schoenberg–Toeplitz matrix  $\mathcal{S}_X(f)$  defines a minimal operator  $S_X(f)$  in  $\ell^2$  if and only if  $f \in L^2(\mathbb{R}_+)$ . In this case  $S_X(f)$  is self-adjoint, its spectrum is purely absolutely continuous and fills in the interval*

$$\sigma(S_X(f)) = \sigma_{ac}(S_X(f)) = [c_-, c_+], \quad 0 < c_\pm = \int_0^\infty \frac{1 \pm e^{-s}}{1 \mp e^{-s}} \tau(ds). \quad (3.40)$$

Moreover, the operator  $S_X(f)$  is bounded if and only if  $f \in L^1(\mathbb{R}_+)$ , or, equivalently,

$$\int_0^\infty \frac{\tau(ds)}{s} < \infty. \quad (3.41)$$

*Proof.* As in the proof of the preceding result, we start with the kernel function  $e_s(u) := e^{-su}$ ,  $s > 0$  and relate the Schoenberg and Toeplitz symbols:

$$a(e_s, e^{i\varphi}) = \sum_{k \in \mathbb{Z}} e^{-s|k|} e^{ik\varphi} = 1 + \frac{e^{-s+i\varphi}}{1 - e^{-s+i\varphi}} + \frac{e^{-s-i\varphi}}{1 - e^{-s-i\varphi}} = \frac{1 - e^{-2s}}{|1 - te^{-s+i\varphi}|^2} = P(e^{-s}, e^{i\varphi}),$$

where  $P(e^{-s}, e^{i\varphi})$  denotes the Poisson kernel for the unit disk. Hence  $S_X(e_s) = \|e^{-s|i-j|}\|_{i,j \in \mathbb{N}}$  is bounded and its spectrum is the interval

$$\sigma(S_X(e_s)) = a(e_s, \mathbb{T}) = \left[ \frac{1 - e^{-s}}{1 + e^{-s}}, \frac{1 + e^{-s}}{1 - e^{-s}} \right].$$

The Toeplitz symbol  $a(f)$  of the operator  $S_X(f) = \|f(|i-j|)\|_{i,j \in \mathbb{N}}$  can be computed as above

$$a(f, e^{i\varphi}) = \sum_{k \in \mathbb{Z}} f(|k|) e^{ik\varphi} = \sum_{k \in \mathbb{Z}} e^{ik\varphi} \int_0^\infty e^{-s|k|} \tau(ds) = \int_0^\infty P(e^{-s}, e^{i\varphi}) \tau(ds), \quad \varphi \neq 0. \quad (3.42)$$

One completes the proof in just the same fashion as in Proposition 3.25. □

Similarly, the condition  $f \in CM_0(\mathbb{R}_+) \cap L^2(\mathbb{R}_+)$  is equivalent to (cf. (3.41))

$$\int_{\mathbb{R}_+^2} \frac{\tau(ds_1) \tau(ds_2)}{s_1 + s_2} < \infty. \quad (3.43)$$

*Example 3.27.* It is not hard to manufacture a Schoenberg–Toeplitz matrices with the Schoenberg symbol  $f \in CM_0(\mathbb{R}_+) \setminus L^2(\mathbb{R}_+)$ . Indeed, one can take

$$\mathcal{S}_X(f_\beta) = \|(1 + |i-j|)^{-\beta}\|_{i,j \in \mathbb{N}}, \quad f_\beta(r) = \frac{1}{(1+r)^\beta} = \frac{1}{\Gamma(\beta)} \int_0^\infty e^{-sr} s^{\beta-1} e^{-s} ds \quad (3.44)$$

with  $0 < \beta \leq 1/2$ . In this example no coordinate vector  $e_j$ ,  $j \in \mathbb{N}$ , belongs to  $\ell^2$ .

*Remark 3.28.* (i). According to a result of Brown and Halmos (see, e.g., [19, Theorem 4.1.4]) the operator  $S_X(f)$  is bounded if and only if  $a(f) \in L^\infty(\mathbb{T})$ . Due to the asymptotic relation (3.38) for  $f \in \Phi_\infty$  the latter is equivalent to (3.36). This observation provides another proof of the last statement of both preceding propositions.

(ii). The relation between the Schoenberg symbol  $f \in \Phi_\infty(\alpha)$  for  $\alpha = 1, 2$  and the Toeplitz symbol  $a(f)$  is implemented by the Poisson kernel and the Jacobi theta-function, respectively. We are unaware of the similar relation for  $1 < \alpha < 2$ .

(iii). A Schoenberg–Toeplitz operator  $S_X(f)$  with  $f \in \mathcal{M}_+$  is bounded if and only if the Fourier coefficients of its Toeplitz symbol  $a(f)$  (3.34) are positive and monotone decreasing and  $a(f) \in W$ , the Wiener algebra of absolutely convergent Fourier series. This result stems directly from Theorem 3.4.

*Example 3.29.* We construct a bounded Schoenberg–Toeplitz operator  $S_X(\varphi)$  with  $0 \in \sigma(S_X(\varphi))$ . Take any Toeplitz sequence  $X \subset \mathbb{R}^1$  so that  $|x_i - x_j| = |i - j|$  and put

$$\varphi(t) = \left(1 - \frac{t}{2}\right)_+ \in \Phi_1, \quad (a)_+ := \max(a, 0).$$

Then  $S_X(\varphi) = J(\{1/2\}, \{1\})$  is the Jacobi operator with 1 on the main diagonal and  $1/2$  off the main diagonal. It is well known that  $\sigma(S_X(\varphi)) = [0, 2]$ , as claimed. Certainly,  $\varphi \notin \Phi_\infty$ .

## 4 Schoenberg matrices and harmonic analysis on $\mathbb{R}^n$

### 4.1 Radial strongly $X$ -positive definite functions

We begin with some basics of harmonic analysis on the Hilbert spaces ([20, Section C.3.3], [34]).

**Definition 4.1.** Let  $\mathcal{F} = \{f_k\}_{k \in \mathbb{N}}$  be a sequence of vectors in a Hilbert space  $\mathcal{H}$ .

- (i)  $\mathcal{F}$  is called a *Riesz–Fischer sequence* if for all  $(\xi_1, \dots, \xi_m) \in \mathbb{C}^m$  and  $m \in \mathbb{N}$  there is a constant  $c > 0$  such that

$$\left\| \sum_{k=1}^m \xi_k f_k \right\|_{\mathcal{H}}^2 \geq c \sum_{k=1}^m |\xi_k|^2. \quad (4.1)$$

- (ii)  $\mathcal{F}$  is called a *Bessel sequence* if for all  $(\xi_1, \dots, \xi_m) \in \mathbb{C}^m$  and  $m \in \mathbb{N}$  there is a constant  $C < \infty$  such that

$$\left\| \sum_{k=1}^m \xi_k f_k \right\|_{\mathcal{H}}^2 \leq C \sum_{k=1}^m |\xi_k|^2. \quad (4.2)$$

- (iii)  $\mathcal{F}$  is called a *Riesz sequence* (or a Riesz basis in its linear span) if  $\mathcal{F}$  is both Riesz–Fischer and Bessel sequence. If  $\mathcal{F}$  is complete we say about a Riesz basis in  $\mathcal{H}$ .

It turns out that the above notions applied to sequences of exponential functions in  $L^2$ -spaces are tightly related to the strong  $X$ -positive definiteness.

Given an arbitrary sequence  $X = \{x_k\}_{k \in \mathbb{N}}$  of distinct points in  $\mathbb{R}^n$ , we introduce a system

$$\mathcal{E}_X = \{e(\cdot, x_k)\}_{k \in \mathbb{N}}, \quad e(x, x_k) = e^{i(x, x_k)}, \quad x \in \mathbb{R}^n, \quad (4.3)$$

of exponential functions.

**Proposition 4.2.** *Let  $g$  be a positive definite function (2.2) with the Bochner measure  $\mu$ . For an arbitrary sequence  $X = \{x_k\}_{k \in \mathbb{N}}$  of distinct points in  $\mathbb{R}^n$  and for the system of exponential functions  $\mathcal{E}_X$  (4.3) the following holds.*

(i)  $\mathcal{E}_X$  is a Riesz–Fischer sequence in  $L^2(\mathbb{R}^n, \mu)$  if and only if  $g$  is strongly  $X$ -positive definite.

(ii)  $\mathcal{E}_X$  is a Bessel sequence if and only if the Gram matrix

$$Gr(\mathcal{E}_X, L^2(\mathbb{R}^n, \mu)) = \|\langle e(\cdot, x_k), e(\cdot, x_j) \rangle_{L^2(\mathbb{R}^n, \mu)}\|_{k,j \in \mathbb{N}} = \|g(x_k - x_j)\|_{k,j \in \mathbb{N}} \quad (4.4)$$

defines a bounded, self-adjoint and nonnegative operator on  $\ell^2$ .

(iii)  $\mathcal{E}_X$  is a Riesz sequence if and only if  $Gr(\mathcal{E}_X, L^2(\mathbb{R}^n, \mu))$  defines a bounded and invertible, nonnegative operator.

*Proof.* It is clear that

$$\sum_{k,j=1}^m g(x_k - x_j) \xi_j \bar{\xi}_k = \int_{\mathbb{R}^n} \left| \sum_{k=1}^m \xi_k e(u, x_k) \right|^2 \mu(du) = \left\| \sum_{k=1}^m \xi_k e(\cdot, x_k) \right\|_{L^2(\mathbb{R}^n, \mu)}^2 \quad (4.5)$$

for  $\xi = \{\xi_1, \dots, \xi_m\} \in \mathbb{C}^m$  and arbitrary  $m \in \mathbb{N}$ . All statements are immediate from (4.5).  $\square$

The same system  $\mathcal{E}$  can be viewed as a system of vectors in another Hilbert space, namely  $L^2(S_r^{n-1})$ ,  $S_r^{n-1}$  is a sphere in  $\mathbb{R}^n$  of radius  $r$ , centered at the origin, with the normalized Lebesgue measure. We denote this system by  $\mathcal{E}_X(S_r^{n-1})$ . Such approach leads to RPDF's (see [13]).

The following result is borrowed from [17, Proposition 2.14]. We present it with the proof because of its importance in the sequel.

**Proposition 4.3.** *Let  $f \in \Phi_n$ ,  $n \geq 2$ , with the measure  $\nu = \nu(f)$  in (1.2). Given an arbitrary sequence  $X = \{x_k\}_{k \in \mathbb{N}}$  of distinct points in  $\mathbb{R}^n$ , the function  $f$  is strongly  $X$ -positive definite if and only if there exists a Borel set  $\mathcal{K} \subset (0, +\infty)$ ,  $\nu(\mathcal{K}) > 0$  such that the system  $\mathcal{E}_X(S_r^{n-1})$  forms a Riesz–Fischer sequence for each  $r \in \mathcal{K}$ . In particular, the function  $f_\rho(\cdot) = \Omega_n(\rho \cdot)$ ,  $\rho > 0$ , is strongly  $X$ -positive definite if and only if the system  $\mathcal{E}_X(S_\rho^{n-1})$  is a Riesz–Fischer sequence.*

*Proof.* It follows from (1.2) and (1.4) that for  $(\xi_1, \dots, \xi_m) \in \mathbb{C}^m$  and  $m \in \mathbb{N}$

$$\sum_{j,k=1}^m f(|x_k - x_j|) \xi_j \bar{\xi}_k = \int_0^{+\infty} \left( \int_{S_r^{n-1}} \left| \sum_{k=1}^m \xi_k e(u, x_k) \right|^2 \sigma_n(du) \right) \nu(dr). \quad (4.6)$$

Suppose that there exists a set  $\mathcal{K}$  as stated above. Then for every  $r \in \mathcal{K}$  there is a constant  $c(r) > 0$  so that

$$\int_{S_r^{n-1}} \left| \sum_{k=1}^m \xi_k e(u, x_k) \right|^2 \sigma_n(du) = \left\| \sum_{k=1}^m \xi_k e(\cdot, x_k) \right\|_{L^2(S_r^{n-1})}^2 \geq c(r) \sum_{k=1}^m |\xi_k|^2. \quad (4.7)$$

Choosing  $c(r)$  bounded and measurable and combining the latter inequality with (4.6), we obtain

$$\sum_{j,k=1}^m f(|x_j - x_k|) \xi_j \bar{\xi}_k \geq \int_{\mathcal{K}} \left( \left\| \sum_{k=1}^m \xi_k e(\cdot, x_k) \right\|_{L_r^2(S^{n-1})}^2 \right) \nu(dr) \geq c \sum_{k=1}^m |\xi_k|^2, \quad (4.8)$$

$$c := \int_{\mathcal{K}} c(r) \nu(dr).$$

Since  $\nu(\mathcal{K}) > 0$  and  $c(r) > 0$ , we have  $c > 0$ , so  $f$  is strongly  $X$ -positive definite.

Conversely, if

$$\int_0^\infty h(r) \nu(dr) \geq c_1 > 0, \quad h(r) = \left\| \sum_{k=1}^m \xi_k e(\cdot, x_k) \right\|_{L_r^2(S^{n-1})}^2,$$

then there is a Borel set  $\mathcal{K} \subset (0, +\infty)$  of positive  $\nu$ -measure such that  $h \geq c_1$  on  $\mathcal{K}$ , as claimed.  $\square$

We want to lay stress on the fact that the measure  $\nu$  enters this result only via existence of a certain Borel set  $\mathcal{K}$  of positive  $\nu$ -measure.

**Corollary 4.4.** *Let  $f_j \in \Phi_n$ ,  $n \geq 2$ ,  $j = 1, 2$ , with the measures  $\nu_1$  and  $\nu_2$  in (1.2), respectively. Assume that  $\nu_1$  is absolutely continuous with respect to  $\nu_2$ . Given a set  $X = \{x_k\}_{k \in \mathbb{N}}$  of distinct points in  $\mathbb{R}^n$ , if  $f_1$  is strongly  $X$ -positive definite then so is  $f_2$ . In particular, if  $\nu_1$  and  $\nu_2$  are mutually absolutely continuous (equivalent), then  $f_1$  and  $f_2$  are strongly  $X$ -positive definite simultaneously.*

*Proof.* By Proposition 4.3 there is a Borel set  $\mathcal{K} \subset (0, +\infty)$ ,  $\nu_1(\mathcal{K}) > 0$  so that the system  $\mathcal{E}_r = \{e(\cdot, rx_k)\}_{k \in \mathbb{N}}$  forms a Riesz–Fischer sequence in  $L^2(S^{n-1})$  for each  $r \in \mathcal{K}$ . Since  $\nu_1$  is absolutely continuous with respect to  $\nu_2$ , then  $\nu_2(\mathcal{K}) > 0$  as well. Now Proposition 4.3 applies in backward direction and yields strong  $X$ -positive definiteness of  $f_2$ , as claimed.  $\square$

We are in a position now to prove the main result of the section.

**Theorem 4.5** (=Theorem 1.6). *Let  $(\text{const} \neq) f \in \Phi_n$ ,  $n \geq 2$ , with the representing measure  $\nu = \nu(f)$  from (1.2). If  $\nu$  is equivalent to the Lebesgue measure on  $\mathbb{R}_+$ , then  $f$  is strongly  $X$ -positive definite for each  $X \in \mathcal{X}_n$ .*

*Proof.* We begin with a function  $f_s(r) := e^{-sr} \in \Phi_n$  and show that for each  $X \in \mathcal{X}_n$   $f_s$  is strongly  $X$ -positive definite for all large enough  $s > 0$ . Indeed, take  $s$  so that

$$\|t^{n-1} f_s\|_{L^1(\mathbb{R}_+)} = \int_0^\infty t^{n-1} e^{-st} dt = \frac{\Gamma(n)}{s^n} < \frac{d_*^n(X)}{5^n n^2}.$$

By Theorem 3.4 (see (3.7)) the Schoenberg operator  $S_X(f_s)$  is bounded and invertible, so (1.11) holds, as needed.

To make use of Corollary 4.4 we compute the measure  $\nu(f_s)$ . To this end recall a well-known result from the Fourier transforms theory, which plays a key role in the sequel.

Let  $h \in L^1(\mathbb{R}^n)$  and let  $\widehat{h}$  be its Fourier transform

$$\widehat{h}(t) := \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} h(x) e^{-i(t,x)} dx.$$

If  $h(\cdot) = h_0(|\cdot|)$  is a radial function, then so is  $\widehat{h}(\cdot) = H_0(|\cdot|)$ . Moreover,  $H_0$  and  $h_0$  are related by (see, e.g., [26, Theorem IV.3.3])

$$H_0(r) = \frac{1}{r^q} \int_0^\infty J_q(ru) u^{q+1} h_0(u) du = \frac{1}{2^q \Gamma(q+1)} \int_0^\infty \Omega_n(ru) u^{n-1} h_0(u) du, \quad q := \frac{n}{2} - 1. \quad (4.9)$$

The latter is usually referred to as the Fourier–Bessel transform.

We apply (4.9) to a pair of functions

$$h(x) = \frac{2^{n/2} \Gamma(\frac{n+1}{2})}{\sqrt{\pi}} \frac{s}{(s^2 + |x|^2)^{\frac{n+1}{2}}}, \quad \widehat{h}(t) = e^{-s|t|},$$

(this is a particular case of (4.24) below) and come to

$$f_s(r) = e^{-sr} = \frac{2}{B(\frac{n}{2}, \frac{1}{2})} \int_0^\infty \Omega_n(ru) \frac{su^{n-1}}{(s^2 + u^2)^{\frac{n+1}{2}}} du, \quad s, t > 0, \quad B(a, b) := \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} \quad (4.10)$$

is the Euler beta-function. This is exactly representation (1.2) of  $f_s$  with the measure

$$\nu = \nu(f_s) = \frac{2}{B(\frac{n}{2}, \frac{1}{2})} \frac{su^{n-1}}{(s^2 + u^2)^{\frac{n+1}{2}}} du,$$

equivalent to the Lebesgue measure. By the assumption of the theorem the measures  $\nu(f)$  and  $\nu(f_s)$  are equivalent. Since  $f_s$  is strongly  $X$ -positive definite for large enough  $s$  and each separated set  $X \in \mathcal{X}_n$ , then by Corollary 4.4, so is  $f$ , as claimed.  $\square$

*Remark 4.6.* In fact, Theorem 4.5 remains valid whenever the Lebesgue measure on  $\mathbb{R}_+$  is absolutely continuous with respect to the measure  $\nu$ , that is,

$$\nu(ds) = \nu_{ac} + \nu_{sing} = \nu'(s) ds + \nu_{sing}, \quad \nu'(s) > 0 \text{ a.e.}, \quad (4.11)$$

$\nu_{sing}$  is a singular measure. This statement is immediate from the obvious identity  $S_X(f) = S_X(f_{ac}) + S_X(f_{sing})$ , where  $f_{ac}$  and  $f_{sing}$  are the  $\Phi_n$ -functions defined by (1.2) with the measures  $\nu_{ac}$  and  $\nu_{sing}$ , respectively. It is also a consequence of Corollary 4.4, applied in its full extent.

**Theorem 4.7** (=Theorem 1.7). *Let  $f \in \Phi_\infty(\alpha)$ ,  $0 < \alpha \leq 2$ , and  $X \in \mathcal{X}_n$ . Then*

(i)  *$f$  is strongly  $X$ -positive definite. In particular, if  $\mathcal{S}_X(f)$  generates an operator  $S_X(f)$  on  $\ell^2$ , then it is positive definite and so invertible.*

(ii) *If the Schoenberg measure  $\sigma = \sigma_f$  in (2.6) satisfies*

$$\int_0^\infty s^{-\frac{d}{\alpha}} \sigma(ds) < \infty, \quad d = \dim \mathcal{L}(X), \quad (4.12)$$

*then the Schoenberg matrix  $\mathcal{S}_X(f)$  generates a bounded (necessarily invertible) operator.*

(iii) *Conversely, let  $S_Y(f)$  be bounded for at least one  $\delta$ -regular set  $Y$ . Then (4.12) holds.*

*Proof.* (i). We apply again (4.9), now to the pair of functions

$$h(x) = (2s)^{-n/2} \exp\left(-\frac{|x|^2}{4s}\right), \quad \widehat{h}(t) = e^{-s|t|^2},$$

to obtain representation (1.2) for  $g_s$

$$g_s(r) := e^{-sr^2} = \frac{1}{2^q \Gamma(q+1)} \int_0^\infty \Omega_n(ru) \frac{u^{n-1}}{(2s)^{n/2}} \exp\left(-\frac{u^2}{4s}\right) du, \quad r, s > 0, \quad (4.13)$$

(cf. [2, Section V.4.3]). Hence for any  $g \in \Phi_\infty$  we can relate integral representations (1.2) and (2.6). Namely, combining (4.13) with (2.6) we arrive at representation (1.2) for  $g \in \Phi_\infty$

$$g(r) = \int_0^\infty \Omega_n(ru) \phi_{n,\sigma}(u) du, \quad \phi_{n,\sigma}(u) = \frac{u^{n-1}}{2^q \Gamma(q+1)} \int_0^\infty (2s)^{-n/2} \exp\left(-\frac{u^2}{4s}\right) \sigma(ds). \quad (4.14)$$

Clearly,  $\nu(g)$  is equivalent to the Lebesgue measure, and the density  $\phi_{n,\sigma}$  is bounded, strictly positive and continuous on  $\mathbb{R}_+$ . The rest is Theorem 4.5.

(ii). By Corollary 3.6, the Schoenberg operator  $S_X(f)$  is bounded. It is invertible in view of the strong  $X$ -positive definiteness of  $f$ .

(iii) is a combination of Theorem 3.4, (iii), and Corollary 3.6. The proof is complete.  $\square$

*Remark 4.8.* As a special case of Theorem 4.7 we get that the function  $g_s$  (see (4.13)) is strongly  $X$ -positive definite for all  $s > 0$  and each  $X \in \mathcal{X}_n$ . The corresponding Schoenberg operator  $S_X(g_s)$  is bounded and invertible by Theorem 4.7.

*Example 4.9.* According to representation (2.6) each  $f \in \Phi_\infty(\alpha)$  is monotone decreasing. The following example demonstrates that the monotonicity is not necessary for  $f$  to be strongly  $X$ -positive definite for each separated set  $X \in \mathcal{X}_n$ . In particular, it gives an example of strongly  $X$ -positive definite function from  $\Phi_n \setminus \Phi_\infty$ .

Let  $K_\mu$  be the modified Bessel function of the second kind and order  $\mu$  (the definition and properties of  $K_\mu$  are given in the next section). By [29, p.435, (5)] the following integral representation holds for  $n \geq 3$

$$h_s(r) := \Omega_n(rs) M_q(rs) = \frac{2(2s)^{n-2}}{B(q, \frac{1}{2})} \int_0^\infty \Omega_n(ru) \frac{u^{n-1}}{(u^4 + 4s^4)^{\frac{n-1}{2}}} du, \quad M_q(t) := \frac{t^q K_q(t)}{2^{q-1} \Gamma(q)} \quad (4.15)$$

is the Whittle–Matérn function, well-established in spatial statistics,  $q = n/2 - 1$ ,  $s > 0$  is a parameter. We show later that  $M_q \in \Phi_\infty$ , so the function  $h_s \in \Phi_n$ . Its representing measure  $\nu(h_s)$  in (1.2) is equivalent to the Lebesgue measure and given explicitly by

$$\nu(h_s) = \frac{2(2s)^{n-2}}{B(q, \frac{1}{2})} \frac{u^{n-1}}{(u^4 + 4s^4)^{\frac{n-1}{2}}} du$$

so by Theorem 4.5  $h_s$  is strongly  $X$ -positive definite function for each  $X \in \mathcal{X}_n$ .

On the other hand,  $h_s$  has infinitely many real zeros, so it is not monotone decreasing and hence  $f \notin \Phi_\infty$ . Thus, by (4.15),  $f \in \Phi_n \setminus \Phi_\infty$ .

*Remark 4.10.* If a real-valued function  $f$  obeys  $|f(r)| \leq ce^{-ar}$ ,  $a > 0$ , (as in the above example), then by Proposition 3.8, the Schoenberg operator  $S_X(f)$  is bounded for each  $X \in \mathcal{X}_n$  and any  $n \in \mathbb{N}$ .

## 4.2 “Grammization” of Schoenberg matrices

Our goal here is to implement the “grammization” procedure, (see Introduction), for two positive definite Schoenberg’s matrices

$$\mathcal{S}_X(f) = \|\exp(-a|x_i - x_j|^2)\|_{i,j \in \mathbb{N}}, \quad \mathcal{S}_X(f) = \|\exp(-a|x_i - x_j|)\|_{i,j \in \mathbb{N}}, \quad a > 0, \quad (4.16)$$

and also for a certain family of Schoenberg’s matrices which contains the second one in (4.16).

The key observation is stated as the following lemma.

**Lemma 4.11.** *Let  $f \in L^2(\mathbb{R}^n)$  and  $f_\xi := f(\cdot - \xi)$  be its shift on  $\xi \in \mathbb{R}^n$ . Then for any  $\xi, \eta \in \mathbb{R}^n$*

$$\langle f_\xi, f_\eta \rangle_{L^2(\mathbb{R}^n)} = \widehat{F}(\xi - \eta), \quad F(t) := (2\pi)^{n/2} |\widehat{f}(t)|^2. \quad (4.17)$$

*Proof.* Since

$$\widehat{f}_\xi(t) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} f_\xi(x) e^{-i(x,t)} dx = \widehat{f}(t) e^{-i(t,\xi)},$$

we have by Parseval’s equality

$$\langle f_\xi, f_\eta \rangle_{L^2(\mathbb{R}^n)} = \langle \widehat{f}_\xi, \widehat{f}_\eta \rangle_{L^2(\mathbb{R}^n)} = \int_{\mathbb{R}^n} |\widehat{f}(t)|^2 e^{-i(t,\xi-\eta)} dt = (2\pi)^{n/2} \widehat{F}(\xi - \eta), \quad \xi, \eta \in \mathbb{R}^n,$$

as claimed.  $\square$

**Proposition 4.12.** *Let  $\xi, \eta \in \mathbb{R}^n$ ,  $a > 0$ . Then*

$$e^{-\frac{a}{2}|\xi-\eta|^2} = \left(\frac{2a}{\pi}\right)^{n/2} \langle h_{a,\xi}, h_{a,\eta} \rangle_{L^2(\mathbb{R}^n)}, \quad h_{a,\xi}(x) = e^{-a|x-\xi|^2}. \quad (4.18)$$

*The grammization of the first Schoenberg’s matrix in (4.16) reads as follows*

$$\|\exp\left(-\frac{a}{2}|x_i - x_j|^2\right)\|_{i,j \in \mathbb{N}} = \left(\frac{2a}{\pi}\right)^{n/2} Gr(\{f_j\}, L^2(\mathbb{R}^n)), \quad f_j(x) = e^{-a|x-x_j|^2}. \quad (4.19)$$

*Proof.* Combining Lemma 4.11 (see (4.17)) with the well-known formula

$$\widehat{e^{-b|\cdot|^2}}(t) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-b|x|^2 - i(x,t)} dx = \frac{1}{(2b)^{n/2}} e^{-\frac{|t|^2}{4b}}, \quad b > 0,$$

yields the result.  $\square$

The grammization of the second Schoenberg matrix in (4.16) is similar but technically more involved.

We begin with the brief reminder of the modified Bessel functions  $K_\mu$  of the second kind of order  $\mu$ , which solve the differential equations

$$t^2 u''(t) + t u'(t) - (t^2 + \mu^2) u(t) = 0, \quad t > 0, \quad \mu \in \mathbb{R}.$$

The asymptotics for  $K_\mu$  is well known (see [1, (9.6.8)–(9.6.9)], [29, p.202, (1)])

$$\begin{aligned} K_\mu(t) &= \begin{cases} \frac{\Gamma(\mu)}{2} \left(\frac{t}{2}\right)^{-\mu} + O(t^{-\mu+2}), & \mu > 0; \\ \log \frac{2}{t} + O(1), & \mu = 0; \end{cases} \quad t \rightarrow 0, \\ K_\mu(t) &= \sqrt{\frac{\pi}{2t}} e^{-t} (1 + O(t^{-1})), \quad t \rightarrow \infty. \end{aligned} \quad (4.20)$$



The functions  $K_\mu$  are known to satisfy  $K_{-\mu} = K_\mu$  and to admit the integral representations (see, e.g., [29, p.172, (4),(5)])

$$\begin{aligned} K_\mu(z) &= \frac{\sqrt{\pi}}{\Gamma(\mu + \frac{1}{2})} \left(\frac{z}{2}\right)^\mu \int_0^\infty e^{-z \cosh r} \sinh^{2\mu}(r) dr \\ &= \frac{\sqrt{\pi}}{\Gamma(\mu + \frac{1}{2})} \left(\frac{z}{2}\right)^\mu \int_1^\infty e^{-zt} (t^2 - 1)^{\mu - \frac{1}{2}} dt, \quad \mu > -\frac{1}{2}, \quad |\arg z| < \frac{\pi}{2}. \end{aligned} \quad (4.21)$$

Clearly,  $K_\mu$  is positive and monotone decreasing function on  $\mathbb{R}_+$ .

**Proposition 4.13.** *Let  $n \geq 2$  and  $K_\mu$  be the modified Bessel function of the second kind of order  $\mu$ ,  $0 \leq \mu < n/4$ . For  $a > 0$  put*

$$f_{a,\mu}(x) := \left(\frac{a}{|x|}\right)^\mu K_\mu(a|x|), \quad f_{a,\mu,\xi}(x) := f_{a,\mu}(x - \xi), \quad x, \xi \in \mathbb{R}^n. \quad (4.22)$$

Then with  $p := \frac{n}{2} - 2\mu > 0$  the following equality holds for all  $\xi, \eta \in \mathbb{R}^n$

$$\left(\frac{|\xi - \eta|}{a}\right)^p K_p(a|\xi - \eta|) = \frac{2^{\frac{n}{2}-2\mu}}{\pi^{\frac{n}{2}} B(\frac{n}{2} - \mu, \frac{1}{2})} \langle f_{a,\mu,\xi}, f_{a,\mu,\eta} \rangle_{L^2(\mathbb{R}^n)}. \quad (4.23)$$

*Proof.* It follows from (4.20) that  $f_{a,\mu} \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$  for  $0 \leq \mu < n/4$ . We begin with the formula for the Fourier transform

$$\widehat{f_{a,\mu}}(t) = \frac{2^{q-\mu} \Gamma(q - \mu + 1)}{(a^2 + |t|^2)^{q-\mu+1}}. \quad (4.24)$$

It is likely to be known, but due to its importance for the sequel, we outline the proof.

As  $f_{a,\mu}$  is a radial function, then so is its Fourier transform  $\widehat{f_{a,\mu}}(\cdot) = F_{a,\mu}(|\cdot|)$  and by (4.9),

$$F_{a,\mu}(r) = \frac{1}{r^q} \int_0^\infty J_q(rs) s^{q+1} f_{a,\mu}(s) ds = \frac{a^\mu}{r^q} \int_0^\infty J_q(rs) K_\mu(as) s^{q-\mu+1} ds, \quad q = \frac{n}{2} - 1.$$

The latter integral is known in the theory of Bessel functions as (see [29, p.410, (1)])

$$\begin{aligned} \int_0^\infty J_q(rs) K_\mu(as) s^{-\lambda} ds &= \frac{\Gamma(\frac{q-\lambda+\mu+1}{2}) \Gamma(\frac{q-\lambda-\mu+1}{2})}{2^{\lambda+1} \Gamma(q+1)} \frac{r^q}{a^{q-\lambda+1}} \times \\ &F\left(\frac{q-\lambda+\mu+1}{2}, \frac{q-\lambda-\mu+1}{2}; q+1; -\frac{r^2}{a^2}\right), \quad q-\lambda+1 > \mu, \end{aligned}$$

$F$  is the Gauss hypergeometric function. The calculation with  $\lambda = \mu - q - 1$  gives

$$q - \lambda + \mu + 1 = 2(q+1), \quad q - \lambda - \mu + 1 = 2(q - \mu + 1) = n - 2\mu > 0,$$

so

$$F_{a,\mu}(r) = 2^{q-\mu} \Gamma(q - \mu + 1) a^{2(\mu-q-1)} F\left(q+1, q - \mu + 1; q+1; -\frac{r^2}{a^2}\right).$$

The known formula for the hypergeometric series

$$F\left(q+1, q - \mu + 1; q+1; -\frac{r^2}{a^2}\right) = \frac{a^{2(q-\mu+1)}}{(a^2 + r^2)^{q-\mu+1}}$$

leads to (4.24).

To apply (4.17) it remains to compute

$$\widehat{F}(t) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} |\widehat{f_{a,\mu}}(u)|^2 e^{-i(t,u)} du = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \frac{e^{-i(t,u)}}{(a^2 + |u|^2)^{2(q-\mu+1)}} du.$$

The latter Fourier transform is known (see, e.g., [31, Theorem 6.13]) and can be computed, for instance, by using again (4.9) and [29, p.434, (2)]

$$g(x) = G(|x|), \quad G(r) = \frac{1}{r^q} \int_0^\infty \frac{s^{q+1} J_q(rs) ds}{(a^2 + s^2)^{2(q-\mu+1)}} = \left(\frac{r}{a}\right)^{q-2\mu+1} \frac{K_{2\mu-q-1}(ar)}{2^{2q-2\mu+1} \Gamma(2(q-\mu+1))}. \quad (4.25)$$

Since

$$q - 2\mu + 1 = \frac{n}{2} - 2\mu = p > 0, \quad K_{-p} = K_p,$$

we have

$$\langle f_{a,\mu,\xi}, f_{a,\mu,\eta} \rangle_{L^2(\mathbb{R}^n)} = \frac{(2\pi)^{n/2} \Gamma^2(\frac{n}{2} - \mu)}{2\Gamma(n - 2\mu)} \left(\frac{|\xi - \eta|}{a}\right)^p K_p(a|\xi - \eta|).$$

But  $\Gamma(2z) = 2^{2z-1} \pi^{-1/2} \Gamma(z) \Gamma(z + 1/2)$  and (4.23) follows.  $\square$

**Corollary 4.14.** *The grammization for the second Schoenberg matrix in (4.16) is*

$$\begin{aligned} \|\exp(-a|x_j - x_k|)\|_{j,k \in \mathbb{N}} &= Gr(\{g_j\}, L^2(\mathbb{R}^n)), \\ g_j(x) &= \sqrt{\frac{2\Gamma(\frac{n+3}{4})a}{\pi^{\frac{n+2}{2}} \Gamma(\frac{n+1}{4})}} \left(\frac{a}{|x - x_j|}\right)^{\frac{n-1}{4}} K_{\frac{n-1}{4}}(a|x - x_j|). \end{aligned} \quad (4.26)$$

In particular,

$$e^{-a|\xi - \eta|} = \frac{a}{2\pi} \int_{\mathbb{R}^3} \frac{e^{-a|x - \xi|}}{|x - \xi|} \frac{e^{-a|x - \eta|}}{|x - \eta|} dx, \quad \xi, \eta \in \mathbb{R}^3, \quad a > 0. \quad (4.27)$$

*Proof.* Take  $\mu = \frac{n-1}{4}$ , so  $p = 1/2$ , and the function in the left side of (4.23) is just the exponential function [29, p.80, (13)]

$$\sqrt{\frac{|\xi - \eta|}{a}} K_{1/2}(a|\xi - \eta|) = \sqrt{\frac{\pi}{2}} \frac{e^{-a|\xi - \eta|}}{a}, \quad (4.28)$$

which is (4.26).

If  $n = 3$ ,  $\mu = 1/2$ , then

$$f_{a,1/2,\xi}(x) = \left(\frac{a}{|x - \xi|}\right)^{1/2} K_{1/2}(a|x - \xi|) = \sqrt{\frac{\pi}{2}} \frac{e^{-a|x - \xi|}}{|x - \xi|}, \quad (4.29)$$

and (4.27) follows.  $\square$

Note that (4.27) is one of the cornerstones of [17] (see formula (3.26) in there).

The case  $n = 2$ ,  $\mu = 0$  leads to the following

**Corollary 4.15.** *For all  $\xi, \eta \in \mathbb{R}^2$  and  $a > 0$*

$$\frac{|\xi - \eta|}{a} K_1(a|\xi - \eta|) = \frac{1}{\pi} \langle K_0(a|\cdot - \xi|), K_0(a|\cdot - \eta|) \rangle_{L^2(\mathbb{R}^2)}.$$

There is another natural way to view (4.23). For arbitrary  $p > 0$  and  $a > 0$  consider the Whittle–Matérn function (cf. (4.15))

$$M_{p,a}(r) := \left(\frac{r}{a}\right)^p K_p(ar), \quad r > 0. \quad (4.30)$$

Since  $K_{-p} = K_p$ , the notation makes sense for negative indices, and another family of the Whittle–Matérn functions comes in

$$\widetilde{M}_{p,a}(r) = M_{-p,a}(r) = \left(\frac{a}{r}\right)^p K_p(ar), \quad p > 0, \quad \widetilde{M}_{0,a}(r) = K_0(r).$$

Then equality (4.23) with  $0 < 2p \leq n$  reads

$$\begin{aligned} M_{p,a}(|\xi - \eta|) &= \langle c_{n,p} \widetilde{M}_{d,a}(|\cdot - \xi|), c_{n,p} \widetilde{M}_{d,a}(|\cdot - \eta|) \rangle_{L^2(\mathbb{R}^n)}, \\ 0 \leq d &:= \frac{1}{2} \left( \frac{n}{2} - p \right) < \frac{n}{4}, \quad c_{n,p}^2 = \frac{2^p}{\pi^{\frac{n}{2}} B(d, \frac{1}{2})} \end{aligned} \quad (4.31)$$

for all  $\xi, \eta \in \mathbb{R}^n$ .

To have a proper normalization at the origin we put (see (4.20) and (4.15))

$$M_p(r) = \frac{M_{p,1}(r)}{2^{p-1}\Gamma(p)} = \frac{r^p K_p(r)}{2^{p-1}\Gamma(p)} = 1 + O(r^2), \quad r \rightarrow 0.$$

As a byproduct of Proposition 4.13 we have (cf. [16], [12, Table 2]).

**Corollary 4.16.**  $M_p \in \Phi_\infty$  for all  $p > 0$ .

*Proof.* Take  $n > 2p$ . By Proposition 4.13, for each finite set  $X \subset \mathbb{R}^n$  the Schoenberg matrix  $\mathcal{S}_X(M_p)$  is the Gramm matrix, so  $\mathcal{S}_X(M_p) \geq 0$ . Hence  $M_p \in \Phi_n$  for all such  $n$ , as claimed.  $\square$

With regard to Corollary 4.16 one might ask whether the functions  $M_p$  belong to certain subclasses of  $\Phi_\infty$ , for instance, to the class  $CM_0(\mathbb{R}_+)$  of completely monotone functions. The result below seems interesting on its own.

**Proposition 4.17.** *For the Whittle–Matérn function  $M_p$  the following statements hold.*

- (i)  $M_p \in CM(\mathbb{R}_+)$  if and only if  $-\infty < p \leq 1/2$ .
- (ii)  $M_p \in CM_0(\mathbb{R}_+)$  if and only if  $0 < p \leq 1/2$ .

*Proof.* The assertion for  $-\infty < p < 1/2$  follows directly from the second integral representation (4.21) and the Bernstein theorem, if one puts  $\nu = -p$ . Note that the Bernstein measure is finite if and only if  $0 < p < 1/2$ . For  $p = 1/2$  we have

$$M_{1/2}(r) = e^{-r} \in CM_0(\mathbb{R}_+).$$

Let now  $p > 1/2$ . We wish to show that inequalities (2.4) are violated for some  $k \geq 1$ . The argument relies on the differentiation formulae for the Bessel functions, which in our notation look as (see [29, p.74])

$$\left(\frac{1}{z} \frac{d}{dz}\right)^m M_{p,1}(z) = (-1)^m M_{p-m,1}(z). \quad (4.32)$$

For  $m = 1$  it displays the fact that  $M_{p,1}$  is monotone decreasing function on  $\mathbb{R}_+$ . For  $m = 2$  we have

$$M_{p,1}''(r) = -M_{p-1,1}(r) + r^2 M_{p-2,1}(r).$$

For  $p \geq 2$  obviously  $r^2 M_{p-2,1} \rightarrow 0$  as  $r \rightarrow +0$ , so  $M_{p,1}''(+0) = -2^{p-2}\Gamma(p-1) < 0$ , which is inconsistent with (2.4) for  $k = 2$ . If  $1 < p < 2$ , then again

$$r^2 M_{p-2,1}(r) = r^p K_{2-p,1}(r) = r^{2p-2} M_{2-p,1}(r) \rightarrow 0, \quad r \rightarrow +0,$$

with the same conclusion.

Finally, let  $1/2 < p < 1$ . From (4.32) with  $m = 1$  one has

$$M_{p,1}'(r) = -r M_{p-1,1}(r) = -r^p K_{1-p}(r) = -r^{2p-1} M_{1-p,1}(r) \rightarrow 0, \quad r \rightarrow +0$$

so  $M_{p,1}'(+0) = 0$  that is impossible for a nonconstant completely monotone function. The proof is complete.  $\square$

*Remark 4.18.* For  $0 \leq p \leq 1/2$  a stronger result is proved in [18], namely,  $e^r M_{p,1}(r) \in CM(\mathbb{R}_+)$ . Our results for the other values of  $p$  seem to be new.

### 4.3 Minimality conditions and Riesz sequences in $L^2(\mathbb{R}^n)$

The classical result of Bari (see, e.g., [9, Theorem 6.2.1], [20, p.170]) states that a sequence  $\{\varphi_k\}_{k \in \mathbb{N}}$  of vectors in a Hilbert space is a Riesz sequence if and only if the corresponding Gramm matrix  $Gr\{\varphi_k\}_{k \in \mathbb{N}}$  generates a bounded and invertible linear operator on  $\ell^2$ . We examine here certain systems of shifted functions from this viewpoint.

The definitions below are standard (cf. [9, Chapter VI]).

**Definition 4.19.** A sequence of vectors  $\{f_j\}_{j \in \mathbb{N}}$  in a Hilbert space  $\mathcal{H}$  is called *minimal*, if neither of  $f_k$  belongs to the closed linear span  $\mathcal{L}(\{f_j\}_{j \neq k})$  of the others. In other words,

$$\delta_k := \text{dist}(f_k / \|f_k\|, \mathcal{L}(\{f_j\}_{j \neq k})) > 0, \quad k \in \mathbb{N}.$$

$\{f_j\}_{j \in \mathbb{N}}$  is *uniformly minimal*, if  $\inf_k \delta_k > 0$ .

Recall that Riesz–Fischer systems are defined in (4.1).

**Lemma 4.20.** Any Riesz–Fischer sequence  $\{f_j\}_{j \in \mathbb{N}}$  is uniformly minimal.

*Proof.* It is clear that a Riesz–Fischer sequence is bounded from below, that is,  $\|f_j\| \geq c$ ,  $j \in \mathbb{N}$ . By Definition 4.1(i), (see (4.1)), for any fix  $j$  and any finite sequence  $\{\xi_k\} \subset \mathbb{C}$

$$\left\| \sum_{k \neq j} \xi_k f_k - f_j \right\|^2 \geq c \left( c + \sum_{k \neq j} |\xi_k|^2 \right) \geq c^2, \quad (4.33)$$

so by Definition 4.19  $\{f_j\}_{j \in \mathbb{N}}$  is uniformly minimal, as claimed.  $\square$

Given a function  $f \in L^2(\mathbb{R}^n)$  and a set  $X = \{x_j\}_{j \in \mathbb{N}} \subset \mathbb{R}^n$ , consider a sequence of the shifted functions  $\mathcal{F}_X(f) = \{f(\cdot - x_j)\}_{j \in \mathbb{N}}$ . Denote  $f_j(\cdot) = f(\cdot - x_j)$ .

**Proposition 4.21** (=Proposition 1.8). Let  $f \in L^2(\mathbb{R}^n)$  be a real-valued and radial function such that  $\hat{f} \neq 0$  a.e. Then the following statements are equivalent.

- (i)  $\mathcal{F}_X(f)$  forms a Riesz–Fischer sequence in  $L^2(\mathbb{R}^n)$ ;
- (ii)  $\mathcal{F}_X(f)$  is uniformly minimal in  $L^2(\mathbb{R}^n)$ ;
- (iii)  $X$  is a separated set, i.e.,  $d_*(X) > 0$ .

*Proof.* Implication (i) $\Rightarrow$ (ii) is immediate from Lemma 4.20.

(ii) $\Rightarrow$ (iii). With no loss of generality we can assume that  $\|f\|_{L^2(\mathbb{R}^n)} = 1$ , so  $\|f_j\|_{L^2(\mathbb{R}^n)} = 1$  for all  $j \in \mathbb{N}$ . The normalization in (4.17) shows that  $\widehat{F}(0) = \|f\|_{L^2(\mathbb{R}^n)}^2 = 1$ .

Let  $\mathcal{F}_X(f)$  be uniformly minimal. Then there exists  $\varepsilon > 0$  such that  $\|f_j - f_k\|^2 \geq 2\varepsilon$  for all  $j \neq k \in \mathbb{N}$ . A combination of the latter inequality with identity (4.17) yields

$$1 - \widehat{F}(|x_j - x_k|) = 1 - \langle f_j, f_k \rangle_{L^2(\mathbb{R}^n)} = \frac{\|f_j - f_k\|^2}{2} \geq \varepsilon, \quad j, k \in \mathbb{N}, \quad (4.34)$$

and so  $d_*(X) > 0$  follows.

(iii) $\Rightarrow$ (i). Let  $d_*(X) > 0$ . As all functions in question are radial, we put

$$F(t) = (2\pi)^{n/2} |\widehat{f}(t)|^2 = F_0(|t|), \quad \widehat{F}(t) = \widetilde{F}_0(|t|). \quad (4.35)$$

Clearly,  $F \geq 0$  a.e. on  $\mathbb{R}^n$  and  $F \in L^1(\mathbb{R}^n)$  since  $f \in L^2(\mathbb{R}^n)$ . Hence, by the inversion formula,

$$\widehat{F}(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-i(t,\xi)} F(t) dt = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{i(t,\xi)} F(t) dt,$$

so  $\widehat{F}$  is a radial positive definite function, i.e.,  $\widetilde{F}_0 \in \Phi_n$ . We see that the measure  $\mu = \mu_{\widehat{F}}$  from the Bochner representation (2.2) of  $\widehat{F}$  is absolutely continuous,  $\mu_{\widehat{F}} = (2\pi)^{-n/2} F dt$ . Moreover, the condition  $\widehat{f} \neq 0$  a.e. implies  $F > 0$  a.e. on  $\mathbb{R}^n$ , that is,  $\mu_{\widehat{F}}$  is equivalent to the Lebesgue measure  $dt$  on  $\mathbb{R}^n$ . Hence, the representing Schoenberg measure  $\nu = \nu_{\widetilde{F}_0}$  from (1.2) is equivalent to the Lebesgue measure on  $\mathbb{R}_+$  due to the relation  $\nu\{[0, r]\} = \mu\{|x| \leq r\}$  between  $\nu$  and  $\mu$ . Thereby the conditions of Theorem 4.5 are met and the function  $\widetilde{F}_0$  is strongly  $X$ -positive definite. By Lemma 4.11 (see identity (4.17)) and Definition 1.5 of strongly  $X$ -positive definite functions, the latter amounts to saying that  $\mathcal{F}_X(f)$  is the Riesz–Fischer system. The proof is complete.  $\square$

Under certain additional assumptions on  $f$  we come to Riesz sequences of the shifted functions.

**Theorem 4.22** (=Theorem 1.9). *Let  $f \in L^2(\mathbb{R}^n)$  be a real-valued and radial function such that its Fourier transform  $\widehat{f} \neq 0$  a.e.. Let  $F$  and  $F_0$  be defined in (4.35) and assume that for some majorant  $h \in \mathcal{M}_+$  (1.8) the relations*

$$|\widetilde{F}_0(s)| \leq h(s), \quad s^{n-1}h(s) \in L^1(\mathbb{R}_+) \quad (4.36)$$

*hold. Then the following statements are equivalent.*

- (i)  $\mathcal{F}_X(f)$  forms a Riesz sequence in  $L^2(\mathbb{R}^n)$ ;
- (ii)  $\mathcal{F}_X(f)$  forms a basis in its linear span;
- (iii)  $\mathcal{F}_X(f)$  is uniformly minimal in  $L^2(\mathbb{R}^n)$ ;
- (iv)  $X$  is a separated set, i.e.,  $d_*(X) > 0$ .

*Proof.* The implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) are obvious. The implication (iii) $\Rightarrow$ (iv) is proved in Proposition 4.21.

It remains to prove that (iv) implies (i). Lemma 4.11 is a key ingredient of the proof. Condition (4.17) now reads

$$Gr(\{f_j, L^2(\mathbb{R}^n)\}) = \mathcal{S}_X(\tilde{F}_0). \quad (4.37)$$

In view of the aforementioned theorem of Bari we need to show that under the hypothesis of Theorem 4.22 the Schoenberg operator  $S_X(\tilde{F}_0)$  is bounded and invertible.

First, assumption (4.36) implies the boundedness of  $S_X(\tilde{F}_0)$  in view of Proposition 3.8, and  $\mathcal{F}_X(f)$  is the Bessel sequence.

Secondly, according to Proposition 4.21, the condition  $\hat{f} \neq 0$  a.e. ensures that  $\mathcal{F}_X(f)$  is the Riesz–Fischer sequence, i.e., the operator  $S_X(\tilde{F}_0)$  is invertible, as claimed. Thus, by (4.37) the Gramm matrix  $Gr(\{f_j, L^2(\mathbb{R}^n)\})$  is bounded and invertible, and the Bari theorem completes the proof.  $\square$

*Remark 4.23.* One can avoid using the Fourier transform when computing  $\tilde{F}_0$  from (4.35) since

$$\hat{F}(t) = \int_{\mathbb{R}^n} f(t+y)f(y)dy = \tilde{F}_0(|t|). \quad (4.38)$$

*Example 4.24.* The conditions of Theorem 4.22 can be verified for the systems we already encountered in the previous section. For instance, as we have seen in Proposition 4.12,

$$f(x) = e^{-a|x|^2} \implies \tilde{F}_0(r) = \left(\frac{1}{4a}\right)^{n/2} e^{-\frac{a}{2}r^2}.$$

Similarly, it is shown in Proposition 4.13 that

$$f(x) = \left(\frac{a}{|x|}\right)^\mu K_\mu(a|x|), \quad 0 \leq \mu < \frac{n}{4} \implies \tilde{F}_0(r) = \frac{B\left(\frac{n}{2} - \mu, \frac{1}{2}\right)}{2^{n-2\mu}} \left(\frac{r}{a}\right)^p K_p(ar), \quad p = \frac{n}{2} - 2\mu.$$

Since in both cases  $\tilde{F}_0 \in \Phi_\infty \subset \mathcal{M}_+$  (cf. Corollary 4.16) and  $\tilde{F}_0$  decays exponentially fast (see (4.20)), Theorem 4.22 applies, so  $\mathcal{F}_X(f)$  is the Riesz sequence for each  $X \in \mathcal{X}_n$ .

In view of applications in the spectral theory let us single out two particular cases of the above example.

**Corollary 4.25.** *Let  $\mathcal{F}_2 = \{K_0(a|\cdot - x_j|)\}_{j \in \mathbb{N}}$  and  $\mathcal{F}_3 = \left\{\frac{e^{-a|\cdot - x_j|}}{|\cdot - x_j|}\right\}_{j \in \mathbb{N}}$ . Then each of the sequences  $\mathcal{F}_2$  and  $\mathcal{F}_3$  forms a Riesz sequence in  $L^2(\mathbb{R}^2)$  and  $L^2(\mathbb{R}^3)$ , respectively, for each  $X \in \mathcal{X}_n$ .*

We show now that a sequence  $\mathcal{F}_X(f)$  can be *minimal* but *not uniformly minimal*, (so necessarily  $d_*(X) = 0$ ), whenever  $\hat{f} \neq 0$  a.e. is replaced by the stronger assumption (4.39). Note that in the following proposition a function  $f$  is not even assumed to be radial.

**Proposition 4.26.** *Given  $f \in L^2(\mathbb{R}^n)$ , assume that its Fourier transform  $\hat{f}$  admits the bound*

$$|\hat{f}(t)| \geq C(1 + |t|)^{-p} \quad (4.39)$$

*for some  $p > 0$ . Then the system  $\mathcal{F}_X(f) = \{f(\cdot - x_j)\}_{j \in \mathbb{N}}$  is minimal in  $L^2(\mathbb{R}^n)$  for each set  $X = \{x_j\}_{j \in \mathbb{N}} \subset \mathbb{R}^n$  with no finite accumulation points.*

*Proof.* Denote  $f_j(\cdot) = f(\cdot - x_j)$ . Since the Fourier transform is a unitary operator in  $L^2(\mathbb{R}^n)$ , the system  $\{f_j\}_{j \in \mathbb{N}}$  is minimal in  $L^2(\mathbb{R}^n)$  if and only if so is the system of their Fourier images  $\{\widehat{f_j}\}_{j \in \mathbb{N}}$ . Note that  $\widehat{f_j} = \widehat{f} e^{-i(\cdot, x_j)}$ ,  $f = f_1$  (recall that  $x_1 = 0$ ). To prove the minimality of  $\{\widehat{f_j}\}_{j \in \mathbb{N}}$ , it suffices (in fact is equivalent) to construct a biorthogonal sequence  $\{h_j\}_{j \in \mathbb{N}}$ ,

$$\langle h_j, \widehat{f_k} \rangle_{L^2(\mathbb{R}^n)} = \int_{\mathbb{R}^n} h_j(t) \overline{\widehat{f}(t)} e^{i(t, x_k)} dt = \delta_{kj}, \quad h_j \in \mathcal{L}(\{\widehat{f_j}\}_{j \in \mathbb{N}}).$$

To this end take a smoothing function  $u$  and its shifts  $u_j$

$$u(x) := \begin{cases} \exp\left(\frac{|x|^2}{|x|^2-1}\right), & |x| \leq 1; \\ 0, & |x| > 1. \end{cases} \quad u_j(x) := u\left(\frac{x - x_j}{\rho_j}\right), \quad \rho_j := \text{dist}(x_j, X \setminus \{x_j\}) > 0$$

for each  $j$ , since  $X$  has no finite accumulation points. By the definition  $u_j(x_k) = \delta_{kj}$ . Since  $u \in C_0^\infty$  (infinitely differentiable with compact support), then both  $u_j$  and  $\widehat{u_j}$  belong to the Schwartz class. Define

$$h_{j,1}(t) := (2\pi)^{-\frac{n}{2}} \frac{\widehat{u_j}(t)}{\widehat{f}(t)} = (2\pi)^{-\frac{n}{2}} \frac{\rho_j^n \widehat{u}(\rho_j t)}{\widehat{f}(t)}.$$

In view of (4.39),  $h_{j,1} \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ , so

$$\langle h_{j,1}, \widehat{f_k} \rangle_{L^2(\mathbb{R}^n)} = \int_{\mathbb{R}^n} h_{j,1}(t) \overline{\widehat{f}(t)} e^{i(t, x_k)} dt = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} \widehat{u_j}(t) e^{i(t, x_k)} dt = u_j(x_k) = \delta_{kj}.$$

We are left with putting  $h_j := \mathbb{P} h_{j,1}$ , where  $\mathbb{P}$  is a projection from  $L^2(\mathbb{R}^n)$  onto  $\mathcal{L}(\{\widehat{f_j}\}_{j \in \mathbb{N}})$ . The proof is complete.  $\square$

It is easy to construct a set  $X$  with  $d_*(X) = 0$ , which has no finite accumulation points.

*Example 4.27.* Let  $f = f_{a,\mu}$  (4.22) with  $0 \leq \mu < n/4$ . Condition (4.39) follows from (4.24), so the system  $\mathcal{F}_X(f)$  is minimal for each set  $X$  of distinct points which has no finite accumulation points.

*Remark 4.28.* Corollary 4.25 is crucial in the study of certain spectral properties of the Schrödinger operator with point interactions [17]. The statement on the system  $\mathcal{F}_3$  was proved in [17, Theorem 3.8] in an absolutely different manner. The appearance of such functions takes its origin in the following classical formulae for the resolvent of the Laplace operator  $H_0 := -\Delta$  in  $\mathbb{R}^3$  and  $\mathbb{R}^2$ , respectively,

$$(H_0 - zI)^{-1} f = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{e^{i\sqrt{z}|x-t|}}{|x-t|} f(t) dt, \quad (H_0 - zI)^{-1} f = \frac{1}{2\pi} \int_{\mathbb{R}^2} K_0(\sqrt{-z}|x-t|) f(t) dt, \quad (4.40)$$

(see [4, formulae (1.1.19), (1.5.15)]). Note also that a special case of Proposition 4.26 regarding minimality of the system  $\mathcal{F}_3$  was proved in [17, Lemma 3.5] in a different manner. In the latter case

$$f(x) = \frac{e^{-a|x|}}{|x|}, \quad \widehat{f}(t) = \sqrt{\frac{2}{\pi}} \frac{1}{a^2 + |t|^2},$$

and (4.39) automatically holds.



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